

OPEN-CLOSED GROMOV-WITTEN INVARIANTS OF 3-DIMENSIONAL CALABI-YAU SMOOTH TORIC DM STACKS

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ABSTRACT. We study open-closed orbifold Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks with respect to Lagrangian branes of Aganagic-Vafa type. We prove an open mirror theorem for toric Calabi-Yau 3-orbifolds, which expresses generating functions of orbifold disk invariants in terms of Abel-Jacobi maps of the mirror curves. This generalizes a conjecture by Aganagic-Vafa and Aganagic-Klemm-Vafa, proved by the first and the second authors, on disk invariants of smooth toric Calabi-Yau 3-folds.

1. INTRODUCTION

Open Gromov-Witten invariants of toric Calabi-Yau 3-folds have been studied extensively by both mathematicians and physicists. They correspond to “A-model topological open string amplitudes” in the physics literature and can be interpreted as intersection numbers of certain moduli spaces of holomorphic maps from bordered Riemann surfaces to the 3-fold with boundaries in a Lagrangian submanifold. The physics prediction of these open Gromov-Witten invariants comes from string dualities: *mirror symmetry* relates the A-model topological string theory of a Calabi-Yau 3-fold X to the B-model topological string theory of the mirror Calabi-Yau 3-fold X^\vee ; the *large N duality* relates the A-model topological string theory of Calabi-Yau 3-folds to the Chern-Simons theory on 3-manifolds.

1.1. Open GW invariants of smooth toric CY 3-folds. Aganagic-Vafa [5] introduce a class of Lagrangian submanifolds in smooth toric Calabi-Yau 3-folds, which are diffeomorphic to $S^1 \times \mathbb{R}^2$. By mirror symmetry, Aganagic-Vafa and Aganagic-Klemm-Vafa [5, 4] relate genus zero open GW invariants (disk invariants) of a smooth toric Calabi-Yau 3-fold X relative to such a Lagrangian submanifold L to the classical Abel-Jacobi map of the mirror Calabi-Yau 3-fold X^\vee , which can be further related to the Abel-Jacobi map to the mirror curve of X . This conjecture is proved in full generality in [27].

By the large N duality, Aganagic-Klemm-Mariño-Vafa propose the topological vertex [3], an algorithm of computing all genera generating functions $F_{\beta', \mu_1, \dots, \mu_h}(\lambda)$ of open Gromov-Witten invariants of (X, L) obtained by fixing a topological type of the map (determined by the degree $\beta' \in H_2(X, L; \mathbb{Z})$ and winding numbers $\mu_1, \dots, \mu_h \in H_1(L; \mathbb{Z}) = \mathbb{Z}$) and summing over the genus of the domain. The algorithm of the topological vertex is proved in full generality in [49].

Bouchard-Klemm-Mariño-Pasquetti propose the remodeling conjecture [7], an algorithm of constructing the B-model topological open string amplitudes in all genera of X^\vee following [47], using Eynard-Orantin’s recursive relation from the theory of matrix models [25]. Combined with the mirror symmetry prediction, this gives an algorithm of computing generating functions $F_{g,h}(Q, X_1, \dots, X_h)$ of open Gromov-Witten invariants of (X, L) obtained by fixing a topological type of the domain (determined by the genus g and number h of boundary circles) and summing over the topological types of the map. The remodeling conjecture is proved in full generality very recently [26].

1.2. Open GW invariants for toric CY 3-orbifolds. There have been attempts to generalize the above results to 3-dimensional Calabi-Yau smooth toric DM stacks. The closed GW theory of orbifolds has been studied for a long time. The physics literature dates back to early 1990s such as [14, 58], which study the quantum cohomology ring of orbifolds. The mathematical definition is given by Chen-Ruan [18] for symplectic orbifolds and by Abramovich-Graber-Vistoli [1, 2] for Deligne-Mumford stacks.

A toric Calabi-Yau 3-orbifold is a 3-dimensional Calabi-Yau smooth toric DM stack with trivial generic stabilizer. The concept of Aganagic-Vafa branes can be extended to the setting of 3-dimensional Calabi-Yau smooth toric DM stacks. These branes are diffeomorphic to $[(S^1 \times \mathbb{R}^2)/G]$ where G is a finite abelian group. The open Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks are defined via localization [53], generalizing the methods in [39]. By localization, open and closed Gromov-Witten invariants of a smooth toric Calabi-Yau 3-fold can be obtained by gluing the Gromov-Witten vertex, a generating function of open Gromov-Witten invariants of \mathbb{C}^3 , which can be reduced to a generating function of certain cubic Hodge integrals [12]. Similarly, open and closed orbifold Gromov-Witten invariants of a 3-dimensional Calabi-Yau smooth toric DM stack can be obtained by gluing the orbifold Gromov-Witten vertex, a generating function of open Gromov-Witten invariants of $[\mathbb{C}^3/G]$ (where G is a finite abelian group acting trivially on $dz_1 \wedge dz_2 \wedge dz_3$), which can be reduced to a generating function of certain cubic abelian Hurwitz-Hodge integrals [53]. The Gromov-Witten vertex has been evaluated in the full 3-leg case [46, 49], but the orbifold Gromov-Witten vertex has only been evaluated in the 1-leg case for $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$ in [61, 62, 54], where $\mathbb{C}^2/\mathbb{Z}_n$ is the A_n surface singularity.

As for mirror symmetry, a mirror theorem for disk invariants of $[\mathbb{C}^3/\mathbb{Z}_4]$ is proved in [10]. The remodeling conjecture is also expected to predict higher genus open Gromov-Witten invariants of toric Calabi-Yau 3-orbifolds via mirror symmetry [7, 8].

1.3. Summary of results. In this paper we study open-closed orbifold Gromov-Witten invariants of a 3-dimensional Calabi-Yau smooth toric DM stack \mathcal{X} relative to a Aganagic-Vafa A-brane \mathcal{L} , and prove a mirror theorem for disk invariants of arbitrary toric Calabi-Yau 3-orbifolds. Open Gromov-Witten invariants of the pair $(\mathcal{X}, \mathcal{L})$ count holomorphic maps from orbicurves to \mathcal{X} with boundaries mapped to \mathcal{L} . Morally, the moduli space of such maps are characterized by the following data:

- topological type (g, h) of the domain orbicurve $(\Sigma, \partial\Sigma)$, where g is the genus and h is the number of boundary holes;
- number of interior marked points n ;
- topological type of the map $u : (\Sigma, \partial\Sigma = \coprod_{i=1}^h R_i) \rightarrow (\mathcal{X}, \mathcal{L})$ given by the degree $\beta' = u_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$ and each $[u_*(R_i)] \in H_1(\mathcal{L}; \mathbb{Z})$, collectively denoted by $\vec{\mu} = ([u_*(R_1)], \dots, [u_*(R_h)])$;
- framing $f \in \mathbb{Z}$ of the Aganagic-Vafa A-brane \mathcal{L} .

We denote this moduli space by $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} | \beta', \vec{\mu})$. If we use the evaluation maps ev_i , $i = 1, \dots, n$ at interior points to pull back classes in $H_{\text{orb}}^*(\mathcal{X})$, we obtain open-closed Gromov-Witten invariants. More precisely, for any framed Aganagic-Vafa brane (\mathcal{L}, f) , where $f \in \mathbb{Z}$ there is a canonically associated orbifold which passes through a torus fixed stacky point $p_\sigma = BG_\sigma$ in \mathcal{X} . Given $\gamma_1, \dots, \gamma_n \in H_{\text{orb}}^*(\mathcal{X}; \mathbb{Q})$, we define open-closed orbifold Gromov-Witten invariant $\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta', \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)}$ via localization using a circle action determined by the framing f ; this is a rational number depending on f and can be viewed as an equivariant invariant. For each topological type (g, h) of the domain bordered Riemann surface, we define a generating function $F_{g,h}^{\mathcal{X}, (\mathcal{L}, f)}$ of open-closed Gromov-Witten invariants which takes value in $H_{\text{orb}}^*(p_\sigma; \mathbb{C})^{\otimes h}$, where $H_{\text{orb}}^*(p_\sigma; \mathbb{C}) = \oplus_{k \in G_\sigma} \mathbb{C} \mathbf{1}_k$. In particular, the disk potential $F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}$ takes values in $H_{\text{orb}}^*(p_\sigma; \mathbb{C})$. In case that \mathcal{L} is an outer brane¹, we denote $\vec{\mu} = (\mu, k) \in H_1(\mathcal{L}; \mathbb{Z}) \subset \mathbb{Z} \times G_\sigma$. The disk potential $F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}$ is the following:

$$F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}(\tau_2, X) = \sum_{\beta', n \geq 0} \sum_{(\mu, k) \in H_1(\mathcal{L}; \mathbb{Z})} \frac{\langle (\tau_2)^n \rangle_{0, \beta', (\mu, k)}^{\mathcal{X}, (\mathcal{L}, f)}}{n!} \cdot X^\mu \mathbf{1}_k.$$

In this paper, we prove a mirror theorem regarding $F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}$ when \mathcal{X} is a semi-projective toric Calabi-Yau 3-orbifold. Mirror symmetry relates the topological A-model string theory to its mirror topological B-model string theory. When the A-model theory is a semi-projective toric Calabi-Yau 3-fold, its mirror is given by a Calabi-Yau hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$ given by a equation $uv = H(x, y, q)$, where $(u, v, x, y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2$ and q is the complex moduli parametrizing the B-model. The function $H(x, y, q)$ is determined by both the combinatorial toric data of \mathcal{X} and the framed brane (\mathcal{L}, f) . The affine curve $H(x, y, q) = 0$ is called the

¹We work with both inner and outer branes. See Section 3.1 for the definition.

mirror curve. Let $y = y(x, q)$ be the solved function from the mirror curve. The B-model disk potential $W_{H,\text{inst}}(x, q)$ is defined to be the power series part of the following Abel-Jacobi integral

$$\int \log y(x, q) \frac{dx}{x} = \text{logarithm part} + W_{H,\text{inst}}(x, q).$$

We define a \mathbb{C} -linear function $L^{\mathcal{L},f} : H_{\text{orb}}^*(p_\sigma; \mathbb{C}) \rightarrow \mathbb{C}$, which sends each 1_k ($k \in G_\sigma$) to a root of unity, and prove a *mirror theorem* for the disk potential:

Theorem 1.1. *Under the open-closed mirror map $\tau = \tau(q)$ and $X = X(\tau, q)$, the complex valued function*

$$L^{\mathcal{L},f}(F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\tau(q), X(x, q))) = W_{H,\text{inst}}(x, q).$$

Remark. There are other open Gromov-Witten invariants relative to different types of Lagrangian submanifolds. Jake Solomon defines open Gromov-Witten invariants of a compact symplectic manifold relative to a Lagrangian submanifold which is the fixed locus of an anti-symplectic involution [55]. The mirror theorem for disk invariants for the quintic 3-fold relative to the real quintic is conjectured in [57] and proved in [51]. It has been generalized to compact Calabi-Yau 3-folds which are projective complete intersections [52], where a mirror theorem for genus one open Gromov-Witten invariants (annulus invariants) is also proved.

Open orbifold Gromov-Witten invariants of compact toric orbifolds with respect to Lagrangian torus fibers have recently been defined in [20], which generalizes the work of [28] on compact toric manifolds. There are mirror theorems on disk invariants in this context (see [16], [17], and [15]).

The rest of the paper is organized as follows. In Section 2 we review the necessary materials concerning toric DM stacks. In Section 3 we apply localizations to relate open-closed Gromov-Witten invariants and descendant Gromov-Witten invariants of 3-dimensional Calabi-Yau toric DM stacks. In Section 4 we prove a mirror theorem for orbifold disk invariants.

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2. SMOOTH TORIC DM STACKS

In this section, we follow the definitions in [37, Section 3.1], with slightly different notation. We work over \mathbb{C} .

2.1. Definition. Let N be a finitely generated abelian group, and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We have a short exact sequence of (additive) abelian groups:

$$0 \rightarrow N_{\text{tor}} \rightarrow N \rightarrow \bar{N} = N/N_{\text{tor}} \rightarrow 0,$$

where N_{tor} is the subgroup of torsion elements in N . Then N_{tor} is a finite abelian group, and $\bar{N} = \mathbb{Z}^n$, where $n = \dim_{\mathbb{R}} N_{\mathbb{R}}$. The natural projection $N \rightarrow \bar{N}$ is denoted $b \mapsto \bar{b}$. A *smooth toric DM stack* is an extension of toric varieties [29, 6]. A smooth toric DM stack is given by the following data:

- vectors $b_1, \dots, b_{r'} \in N$. We require the subgroup $\oplus_{i=1}^{r'} \mathbb{Z}b_i$ is of finite index in N .
- a simplicial fan Σ in $N_{\mathbb{R}}$ such that the set of 1-cones is

$$\{\rho_1, \dots, \rho_{r'}\},$$

where $\rho_i = \mathbb{R}_{\geq 0}b_i$, $i = 1, \dots, r'$.

The datum $\Sigma = (\Sigma, (b_1, \dots, b_{r'}))$ is the *stacky fan* in the sense of [6]. The vectors $b_1, \dots, b_{r'}$ may not generate N . We choose *additional* vectors $b_{r'+1}, \dots, b_r$ such that b_1, \dots, b_r generate N . There is a surjective group homomorphism

$$\begin{aligned} \phi: \quad \tilde{N} := \oplus_{i=1}^r \mathbb{Z}\tilde{b}_i &\longrightarrow N, \\ \tilde{b}_i &\longmapsto b_i. \end{aligned}$$

Define $\mathbb{L} := \text{Ker}(\phi) \cong \mathbb{Z}^k$, where $k := r - n$. Then we have the following short exact sequence of finitely generated abelian groups:

$$(1) \quad 0 \rightarrow \mathbb{L} \xrightarrow{\psi} \tilde{N} \xrightarrow{\phi} N \rightarrow 0.$$

Applying $-\otimes_{\mathbb{Z}} \mathbb{C}^*$ to (1), we obtain an exact sequence of abelian groups:

$$(2) \quad 1 \rightarrow K \rightarrow G \rightarrow \widetilde{\mathbb{T}} \rightarrow \mathbb{T} \rightarrow 1,$$

where

$$\begin{aligned} \mathbb{T} &:= N \otimes_{\mathbb{Z}} \mathbb{C}^* = \bar{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n, \\ \widetilde{\mathbb{T}} &:= \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^r, \\ G &:= \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^k, \\ K &:= \mathrm{Tor}_1^{\mathbb{Z}}(N, \mathbb{C}^*) \cong N_{\mathrm{tor}}. \end{aligned}$$

The action of $\widetilde{\mathbb{T}}$ on itself extends to a $\widetilde{\mathbb{T}}$ -action on $\mathbb{C}^r = \mathrm{Spec} \mathbb{C}[Z_1, \dots, Z_r]$. G acts on \mathbb{C}^r via the group homomorphism $G \rightarrow \widetilde{\mathbb{T}}$ in (2), so $K \subset G$ acts on \mathbb{C}^r trivially.

With the above preparation, we are now ready define the smooth toric DM stack \mathcal{X} . Let

$$\mathcal{A} = \{I \subset \{1, \dots, r\} : \sum_{i \notin I} \mathbb{R}_{\geq 0} b_i \text{ is a cone of } \Sigma\}$$

be the set of anti-cones. Given $I \in \mathcal{A}$, let \mathbb{C}^I be the subvariety of \mathbb{C}^r defined by the ideal in $\mathbb{C}[Z_1, \dots, Z_r]$ generated by $\{Z_i \mid i \in I\}$. Define the smooth toric DM stack \mathcal{X} as the stack quotient

$$\mathcal{X} := [U_{\mathcal{A}}/G],$$

where

$$U_{\mathcal{A}} := \mathbb{C}^r \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I.$$

\mathcal{X} contains the DM torus $\mathcal{T} := [\widetilde{\mathbb{T}}/G]$ as a dense open subset, and the $\widetilde{\mathbb{T}}$ -action on \mathcal{U}_{Σ} descends to a \mathcal{T} -action on \mathcal{X} . The smooth toric DM stack \mathcal{X} is a *toric orbifold* if the G -action on $\widetilde{\mathbb{T}}$ is free.

Remark 2.1. The purpose of introducing additional vector $b_{r'+1}, \dots, b_r$ is to ensure G is a *connected* torus. The stacky fan Σ together with the extra vectors $b_{r'+1}, \dots, b_r$ is an *extended stacky fan* in the sense of Jiang [38]. It follows from the definition that $\{r'+1, \dots, r\} \subset I$ for any $I \in \mathcal{A}$. An element of \mathcal{A} is usually referred as an “anti-cone”.

We introduce the following character lattices:

$$\begin{aligned} M &= \mathrm{Hom}(N, \mathbb{Z}) = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*), \\ \widetilde{M} &= \mathrm{Hom}(\widetilde{N}, \mathbb{Z}) = \mathrm{Hom}(\widetilde{\mathbb{T}}, \mathbb{C}^*), \\ \mathbb{L}^{\vee} &= \mathrm{Hom}(\mathbb{L}, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{C}^*). \end{aligned}$$

Applying $\mathrm{Hom}(-, \mathbb{Z})$ to (1), we obtain the following exact sequence of (additive) abelian groups:

$$0 \rightarrow M \xrightarrow{\phi^{\vee}} \widetilde{M} \xrightarrow{\psi^{\vee}} \mathbb{L}^{\vee} \rightarrow \mathrm{Ext}^1(N, \mathbb{Z}) \rightarrow 0$$

Therefore, the group homomorphism

$$\psi^{\vee} : \widetilde{M} \rightarrow \mathbb{L}^{\vee}$$

is surjective if and only if $N_{\mathrm{tor}} = 0$.

We now consider a class of examples of 3-dimensional Calabi-Yau smooth toric DM stacks of the form $[\mathbb{C}^3/\mathbb{Z}_3]$. Let $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$ be the generator of \mathbb{Z}_3 . Given $i, j, k \in \{0, 1, 2\}$ such that $i + j + k \in 3\mathbb{Z}$, we define $\mathcal{X}_{i,j,k}$ to be the quotient stack of the following \mathbb{Z}_3 -action on \mathbb{C}^3 :

$$\omega \cdot (Z_1, Z_2, Z_3) = (\omega^i Z_1, \omega^j Z_2, \omega^k Z_3).$$

In the following example, we consider

$$\mathcal{X}_{1,1,1}, \quad \mathcal{X}_{1,2,0} = [\mathbb{C}^2/\mathbb{Z}_3] \times \mathbb{C}, \quad \mathcal{X}_{0,0,0} = \mathbb{C}^3 \times B\mathbb{Z}_3.$$

Example 2.2. (1) $\mathcal{X} = \mathcal{X}_{1,1,1}$. The toric data are given as follows.

$$\begin{aligned}
N &= \mathbb{Z}^3, \quad N_{\text{tor}} = 0; \\
b_1 &= (1, 0, 1), b_2 = (0, 1, 1), b_3 = (-1, -1, 1), b_4 = (0, 0, 1); \\
r &= 4, r' = 3, k = 1; \\
\Sigma &= \{\text{the 3-cone spanned by } \{b_1, b_2, b_3\}, \text{ and its faces, and faces of faces, etc.}\}; \\
\mathcal{A} &= \{I \subset \{1, 2, 3, 4\} : 4 \in I\}; \\
\mathbb{L} &\cong \mathbb{Z}, \quad \mathbb{L}^\vee \cong \mathbb{Z};
\end{aligned}$$

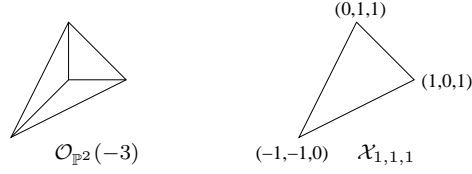


FIGURE 1. $\mathcal{X}_{1,1,1}$ and its crepant resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$

(2) $\mathcal{X} = \mathcal{X}_{1,2,0}$, transversal A_2 -singularity. The toric data are given as follows.

$$\begin{aligned}
N &= \mathbb{Z}^3, \quad N_{\text{tor}} = 0; \\
b_1 &= (1, 0, 1), b_2 = (0, 3, 1), b_3 = (0, 0, 1), b_4 = (0, 1, 1), b_5 = (0, 2, 1); \\
r &= 5, r' = 3, k = 2; \\
\Sigma &= \{\text{the 3-cone spanned by } \{b_1, b_2, b_3\}, \text{ and its faces, and faces of faces, etc.}\}; \\
\mathcal{A} &= \{I \subset \{1, 2, 3, 4, 5\} : \{4, 5\} \subset I\}; \\
\mathbb{L} &\cong \mathbb{Z}^2, \quad \mathbb{L}^\vee \cong \mathbb{Z}^2.
\end{aligned}$$

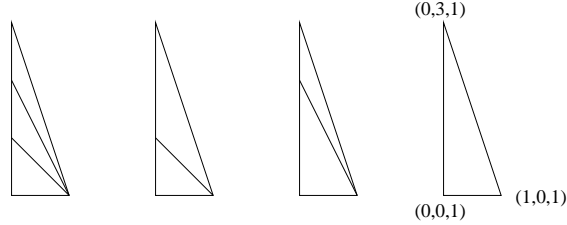


FIGURE 2. $\mathcal{X}_{1,2,0}$ and its (partial) crepant resolutions

(3) $\mathcal{X} = \mathcal{X}_{0,0,0}$. The toric data is given as follows.

$$\begin{aligned}
N &= \mathbb{Z}^3 \oplus \mathbb{Z}_3, \quad N_{\text{tor}} = \mathbb{Z}_3; \\
b_1 &= (1, 0, 0, 0), b_2 = (0, 1, 0, 0), b_3 = (0, 0, 1, 0), b_4 = (1, 0, 0, 1); \\
r &= 4, r' = 3, k = 1; \\
\Sigma &= \{\text{the 3-cone spanned by } \{b_1, b_2, b_3\}, \text{ and its faces, and faces of faces, etc.}\}; \\
\mathcal{A} &= \{I \subset \{1, 2, 3, 4\} : 4 \in I\}; \\
\mathbb{L} &\cong \mathbb{Z}, \quad \mathbb{L}^\vee \cong \mathbb{Z}.
\end{aligned}$$

2.2. Equivariant line bundles and torus-invariant Cartier divisors. A character $\chi \in \widetilde{M}$ gives a $\widetilde{\mathbb{T}}$ -action on $\mathbb{C}^r \times \mathbb{C}$ by

$$(\tilde{t}_1, \dots, \tilde{t}_r) \cdot (Z_1, \dots, Z_r, u) = (\tilde{t}_1 Z_1, \dots, \tilde{t}_r Z_r, \chi(\tilde{t}_1, \dots, \tilde{t}_r)u),$$

where

$$(\tilde{t}_1, \dots, \tilde{t}_r) \in \widetilde{\mathbb{T}} \cong (\mathbb{C}^*)^r, \quad (Z_1, \dots, Z_r) \in \mathbb{C}^r, \quad u \in \mathbb{C}.$$

Therefore $\mathbb{C}^r \times \mathbb{C}$ can be viewed as the total space of a $\widetilde{\mathbb{T}}$ -equivariant line bundle \widetilde{L}_χ over \mathbb{C}^r . If

$$\chi(\tilde{t}_1, \dots, \tilde{t}_r) = \prod_{i=1}^r \tilde{t}_i^{c_i},$$

where $c_1, \dots, c_r \in \mathbb{Z}$, then

$$\widetilde{L}_\chi = \mathcal{O}_{\mathbb{C}^r}(\sum_{i=1}^r c_i \widetilde{D}_i),$$

where \widetilde{D}_i is the $\widetilde{\mathbb{T}}$ -divisor in \mathbb{C}^r defined by $Z_i = 0$. We have

$$\widetilde{M} \cong \text{Pic}_{\widetilde{\mathbb{T}}}(\mathbb{C}^r) \cong H_{\widetilde{\mathbb{T}}}^2(\mathbb{C}^r; \mathbb{Z}),$$

where the first isomorphism is given by $\chi \mapsto \widetilde{L}_\chi$ and the second isomorphism is given by the $\widetilde{\mathbb{T}}$ -equivariant first Chern class $(c_1)_{\widetilde{\mathbb{T}}}$. Define

$$D_i^{\mathcal{T}} := (c_1)_{\widetilde{\mathbb{T}}}(\mathcal{O}_{\mathbb{C}^r}(\widetilde{D}_i)) \in H_{\widetilde{\mathbb{T}}}^2(\mathbb{C}^r; \mathbb{Z}) \cong H_{\mathcal{T}}^2([\mathbb{C}^r/G]; \mathbb{Z}).$$

Then $\{D_1^{\mathcal{T}}, \dots, D_r^{\mathcal{T}}\}$ is a \mathbb{Z} -basis of $H_{\widetilde{\mathbb{T}}}^2(\mathbb{C}^r; \mathbb{Z}) \cong \widetilde{M}$ dual to the \mathbb{Z} -basis $\{\tilde{b}_1, \dots, \tilde{b}_r\}$ of \widetilde{N} . We have a commutative diagram

$$\begin{array}{ccccc} \text{Pic}_{\widetilde{\mathbb{T}}}(\mathbb{C}^r) & \xrightarrow{\iota_{\mathcal{T}}^*} & \text{Pic}_{\widetilde{\mathbb{T}}}(U_{\mathcal{A}}) & \xrightarrow{\cong} & \text{Pic}_{\mathcal{T}}(\mathcal{X}) \\ (c_1)_{\widetilde{\mathbb{T}}} \downarrow & & (c_1)_{\widetilde{\mathbb{T}}} \downarrow & & (c_1)_{\mathcal{T}} \downarrow \\ H_{\widetilde{\mathbb{T}}}^2(\mathbb{C}^r; \mathbb{Z}) & \xrightarrow{\iota_{\mathcal{T}}^*} & H_{\widetilde{\mathbb{T}}}^2(U_{\mathcal{A}}; \mathbb{Z}) & \xrightarrow{\cong} & H_{\mathcal{T}}^2(\mathcal{X}; \mathbb{Z}), \end{array}$$

where $\iota_{\mathcal{T}}^*$ is a surjective group homomorphism induced by the inclusion $\iota : U_{\mathcal{A}} \hookrightarrow \mathbb{C}^r$, and

$$\text{Ker}(\iota_{\mathcal{T}}^*) = \bigoplus_{i=r'+1}^r \mathbb{Z} D_i^{\mathcal{T}}$$

Therefore,

$$\text{Pic}_{\mathcal{T}}(\mathcal{X}) \cong H_{\mathcal{T}}^2(\mathcal{X}; \mathbb{Z}) \cong \widetilde{M} / \bigoplus_{i=r'+1}^r \mathbb{Z} D_i^{\mathcal{T}}$$

Let $\bar{D}_i^{\mathcal{T}} := \iota_{\mathcal{T}}^* D_i^{\mathcal{T}}$. Then

$$\bar{D}_i^{\mathcal{T}} = 0, \quad i = r' + 1, \dots, r,$$

and

$$H_{\mathcal{T}}^2(\mathcal{X}; \mathbb{Z}) = \bigoplus_{i=1}^{r'} \mathbb{Z} \bar{D}_i^{\mathcal{T}} \cong \mathbb{Z}^{r'}.$$

For $i = 1, \dots, r'$, $\widetilde{D}_i \cap U_{\mathcal{A}}$ is a $\widetilde{\mathbb{T}}$ -divisor in $U_{\mathcal{A}}$, and it descends to a \mathbb{T} -divisor \mathcal{D}_i in \mathcal{X} . We have

$$\bar{D}_i^{\mathcal{T}} = (c_1)_{\mathcal{T}}(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)), \quad i = 1, \dots, r'.$$

For $i = r' + 1, \dots, r$, $\widetilde{D}_i \cap U_{\mathcal{A}}$ is empty, so its the zero $\widetilde{\mathbb{T}}$ -divisor.

2.3. Line bundles and Cartier divisors. We have group isomorphisms

$$\mathbb{L}^\vee \cong \text{Pic}_G(\mathbb{C}^r) \cong H_G^2(\mathbb{C}^r; \mathbb{Z}),$$

where the first isomorphism is given by $\chi \in \mathbb{L}^\vee = \text{Hom}(G, \mathbb{C}^*) \mapsto \tilde{L}_\chi$, and the second isomorphism is given by the G -equivariant first Chern class $(c_1)_G$. We have a commutative diagram

$$\begin{array}{ccccc} \text{Pic}_G(\mathbb{C}^r) & \xrightarrow{\iota^*} & \text{Pic}_G(U_{\mathcal{A}}) & \xrightarrow{\cong} & \text{Pic}(\mathcal{X}) \\ (c_1)_G \downarrow & & (c_1)_G \downarrow & & c_1 \downarrow \\ H_G^2(\mathbb{C}^r; \mathbb{Z}) & \xrightarrow{\iota^*} & H_G^2(U_{\mathcal{A}}; \mathbb{Z}) & \xrightarrow{\cong} & H^2(\mathcal{X}; \mathbb{Z}), \end{array}$$

where ι^* is a surjective group homomorphism induced by the inclusion $\iota : U_{\mathcal{A}} \hookrightarrow \mathbb{C}^r$. The surjective map $H_G^2(\mathbb{C}^r; \mathbb{Z}) \rightarrow H^2(\mathcal{X}; \mathbb{Z})$ is the restriction of the Kirwan map

$$\kappa : H_G^*(\mathbb{C}^r; \mathbb{Z}) \longrightarrow H^*(\mathcal{X}; \mathbb{Z}).$$

Define

$$D_i := (c_1)_G(\mathcal{O}_{\mathbb{C}^r}(\tilde{D}_i)) \in H_G^2(\mathbb{C}^r; \mathbb{Z}) \cong H^2([\mathbb{C}^r/G]; \mathbb{Z}).$$

Then

$$\text{Ker}(\iota^*) = \bigoplus_{i=r'+1}^r \mathbb{Z}D_i.$$

Therefore,

$$\text{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{L}^\vee / \oplus_{i=r'+1}^r \mathbb{Z}D_i.$$

Recall that

$$\psi^\vee : \widetilde{M} \rightarrow \mathbb{L}^\vee$$

is surjective if and only if $N_{\text{tor}} = 0$. Let

$$\bar{D}_i = c_1(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)) \in H^2(\mathcal{X}; \mathbb{Z}), \quad i = 1, \dots, r.$$

The map

$$\bar{\psi}^\vee : \text{Pic}_{\mathcal{T}}(\mathcal{X}) \cong H_{\mathcal{T}}^2(\mathcal{X}; \mathbb{Z}) \rightarrow \text{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}),$$

given by

$$\bar{D}_i^{\mathcal{T}} \mapsto \bar{D}_i, \quad i = 1, \dots, r',$$

is surjective if and only if $N_{\text{tor}} = 0$. In general, $\text{Coker}(\psi^\vee) \cong \text{Coker}(\bar{\psi}^\vee)$ is a finite abelian group.

Pick a \mathbb{Z} -basis $\{e_1, \dots, e_k\}$ of $\mathbb{L} \cong \mathbb{Z}^k$, and let $\{e_1^\vee, \dots, e_k^\vee\}$ be the dual \mathbb{Z} -basis of \mathbb{L}^\vee . For each $a \in \{1, \dots, k\}$, we define a *charge vector*

$$l^{(a)} = (l_1^{(a)}, \dots, l_r^{(a)}) \in \mathbb{Z}^r$$

by

$$\psi(e_a) = \sum_{i=1}^r l_r^{(a)} \tilde{b}_r,$$

where $\psi : L \rightarrow \widetilde{M}$ is the inclusion map. Then

$$\psi^\vee(D_i) = \sum_{a=1}^k l_r^{(a)} e_a^*, \quad i = 1, \dots, r,$$

and

$$\sum_{i=1}^r l_r^{(a)} b_r = \phi \circ \psi(e_a) = 0, \quad a = 1, \dots, k.$$

Example 2.3. We use the notation in Example 2.2.

(1) $\mathcal{X} = \mathcal{X}_{1,1,1}$.

$$\begin{aligned} D_1 = D_2 = D_3 = 1, \quad D_4 = -3; \\ l^{(1)} = (1, 1, 1, -3); \\ \text{Pic}_{\mathcal{T}}(\mathcal{X}) \cong \mathbb{Z}^3, \quad \text{Pic}(\mathcal{X}) \cong \mathbb{Z}/3\mathbb{Z}; \end{aligned}$$

(2) $\mathcal{X} = \mathcal{X}_{1,2,0}$.

$$\begin{aligned} D_1 = (0, 0), \quad D_2 = (0, 1), \quad D_3 = (1, 0), \quad D_4 = (-2, 1), \quad D_5 = (1, -2); \\ l^{(1)} = (0, 0, 1, -2, 1), \quad l^{(2)} = (0, 1, 0, 1, -2); \\ \text{Pic}_{\mathcal{T}}(\mathcal{X}) = \mathbb{Z}^3, \quad \text{Pic}(\mathcal{X}) = \mathbb{Z}^2 / (\mathbb{Z}(-2, 1) \oplus \mathbb{Z}(1, -2)) \cong \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

(3) $\mathcal{X} = \mathcal{X}_{0,0,0}$.

$$\begin{aligned} D_1 = 3, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = -3; \\ l^{(1)} = (3, 0, 0, -3); \\ \text{Pic}_{\mathcal{T}}(\mathcal{X}) = \mathbb{Z}^3, \quad \text{Pic}(\mathcal{X}) = \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

2.4. Torus invariant subvarieties and their generic stabilizers. Let $\Sigma(d)$ be the set of d -dimensional cones. For each $\sigma \in \Sigma(d)$, define

$$I_{\sigma} = \{i \in \{1, \dots, r\} \mid \rho_i \not\subset \sigma\} \in \mathcal{A},$$

and define

$$I'_{\sigma} = \{1, \dots, r\} \setminus I_{\sigma}.$$

Then $|I'_{\sigma}| = d$ and $|I_{\sigma}| = r - d$. Let $\tilde{V}(\sigma) \subset U_{\mathcal{A}}$ be the closed subvariety defined by the ideal of $\mathbb{C}[Z_1, \dots, Z_r]$ generated by

$$\{Z_i = 0 \mid \rho_i \subset \sigma\} = \{Z_i = 0 \mid i \in I'_{\sigma}\}.$$

Then $\mathcal{V}(\sigma) := [\tilde{V}(\sigma)/G]$ is an $(n - d)$ -dimensional \mathcal{T} -invariant closed subvariety of $\mathcal{X} = [U_{\mathcal{A}}/G]$.

The group homomorphism $G \cong (\mathbb{C}^*)^k \rightarrow \tilde{\mathbb{T}} \cong (\mathbb{C}^*)^r$ is given by

$$g \mapsto (\chi_1(g), \dots, \chi_r(g)),$$

where $\chi_i \in \text{Hom}(G, \mathbb{C}^*) = \mathbb{L}^{\vee}$ is given by

$$\chi_i(u_1, \dots, u_k) = \prod_{a=1}^k u_a^{l_i^{(a)}}.$$

Let

$$G_{\sigma} := \{g \in G \mid g \cdot z = z \text{ for all } z \in \tilde{V}(\sigma)\} = \bigcap_{i \in I_{\sigma}} \text{Ker}(\chi_i).$$

Then G_{σ} is the generic stabilizer of $\mathcal{V}(\sigma)$. It is a finite subgroup of G . If $\tau \subset \sigma$ then $I_{\sigma} \subset I_{\tau}$, so $G_{\tau} \subset G_{\sigma}$. There are two special cases:

- Let $\{0\}$ be the unique 0-dimensional cone. Then $G_{\{0\}} = K$ is the generic stabilizer of $\mathcal{V}(\{0\}) = \mathcal{X}$.
- If $\sigma \in \Sigma(r)$, then $\mathfrak{p}_{\sigma} := \mathcal{V}(\sigma)$ is a \mathcal{T} fixed point in \mathcal{X} , and $\mathfrak{p}_{\sigma} = BG_{\sigma}$.

Example 2.4. We use the notation in Example 2.2. Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^3$ denote the 3-dimensional cone spanned by $\bar{b}_1, \bar{b}_2, \bar{b}_3$. For $j = 1, 2, 3$, let τ_j denote the 2-dimensional cone in $N_{\mathbb{R}}$ spanned by $\{\bar{b}_i : i \in \{1, 2, 3\} - \{j\}\}$.

- (1) $\mathcal{X} = \mathcal{X}_{1,1,1}$: $G_{\sigma} = \mathbb{Z}_3$, $G_{\tau_1} = G_{\tau_2} = G_{\tau_3} = \{1\}$.
- (2) $\mathcal{X} = \mathcal{X}_{1,2,0}$: $G_{\sigma} = \mathbb{Z}_3 = G_{\tau_3}$, $G_{\tau_1} = G_{\tau_2} = \{1\}$.
- (3) $\mathcal{X} = \mathcal{X}_{0,0,0}$: $G_{\sigma} = \mathbb{Z}_3 = G_{\tau_1} = G_{\tau_2} = G_{\tau_3}$.

Define the set of flags in Σ to be

$$F(\Sigma) = \{(\tau, \sigma) \in \Sigma(r-1) \times \Sigma(r) : \tau \subset \sigma\}.$$

Given $(\tau, \sigma) \in F(\Sigma)$, let $\mathfrak{l}_\tau := \widetilde{V}(\tau)$ be the 1-dimensional \widetilde{T} -invariant subvariety of \mathcal{X} . Then \mathfrak{p}_σ is contained in \mathfrak{l}_τ . There is a unique $i \in \{1, \dots, r'\}$ such that $i \in I'_\sigma \setminus I'_\tau$. The representation of G_σ on the tangent line $T_{\mathfrak{p}_\sigma} \mathfrak{l}_\tau$ to \mathfrak{l}_τ at the stacky point \mathfrak{p}_σ is given by $\chi_i|_{G_\sigma} : G_\sigma \rightarrow \mathbb{C}^*$. The image $\chi_i(G_\sigma) \subset \mathbb{C}^*$ is a cyclic subgroup of \mathbb{C}^* ; we define the order of this group to be $r(\tau, \sigma)$. Then there is a short exact sequence of finite abelian groups:

$$1 \rightarrow G_\tau \rightarrow G_\sigma \rightarrow \mu_{r(\tau, \sigma)} \rightarrow 1,$$

where μ_a is the group of a -th roots of unity.

2.5. The extended nef cone and the extended Mori cone. In this paragraph, $\mathbb{F} = \mathbb{Q}, \mathbb{R},$ or \mathbb{C} . Given a finitely generated abelian group Λ with $\Lambda/\Lambda_{\text{tor}} \cong \mathbb{Z}^m$, define $\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F} \cong \mathbb{F}^m$. We have the following short exact sequences of vector spaces:

$$\begin{aligned} 0 \rightarrow \mathbb{L}_{\mathbb{F}} \rightarrow \widetilde{N}_{\mathbb{F}} \rightarrow N_{\mathbb{F}} \rightarrow 0, \\ 0 \rightarrow M_{\mathbb{F}} \rightarrow \widetilde{M}_{\mathbb{F}} \rightarrow \mathbb{L}_{\mathbb{F}}^{\vee} \rightarrow 0. \end{aligned}$$

We also have the following isomorphisms of vector spaces over \mathbb{F} :

$$\begin{aligned} H^2(\mathcal{X}; \mathbb{F}) &\cong H^2(X; \mathbb{F}) \cong \mathbb{L}_{\mathbb{F}}^{\vee} / \oplus_{i=r'+1}^r \mathbb{F} D_i, \\ H_{\mathcal{T}}^2(\mathcal{X}; \mathbb{F}) &\cong H_{\mathbb{T}}^2(X; \mathbb{F}) \cong \widetilde{M}_{\mathbb{F}} / \oplus_{i=r'+1}^r \mathbb{F} D_i^{\mathcal{T}}, \end{aligned}$$

where X is the coarse moduli space of \mathcal{X} .

From now on, we assume all the maximal cones in Σ are n -dimensional, where $n = \dim_{\mathbb{C}} \mathcal{X}$. Given a maximal cone $\sigma \in \Sigma(n)$, we define

$$\mathbb{K}_{\sigma}^{\vee} := \bigoplus_{i \in I_{\sigma}} \mathbb{Z} D_i.$$

Then $\mathbb{K}_{\sigma}^{\vee}$ is a sublattice of \mathbb{L}^{\vee} of finite index. We define the *extended σ -nef cone* to be

$$\widetilde{\text{Nef}}_{\sigma} = \sum_{i \in I_{\sigma}} \mathbb{R}_{\geq 0} D_i,$$

which is a k -dimensional cone in $\mathbb{L}_{\mathbb{R}}^{\vee} \cong \mathbb{R}^k$. The *extended nef cone* of the extended stacky fan $(\Sigma, b_1, \dots, b_r)$ is

$$\widetilde{\text{Nef}}_{\mathcal{X}} := \bigcap_{\sigma \in \Sigma(n)} \widetilde{\text{Nef}}_{\sigma}.$$

The *extended σ -Kähler cone* \widetilde{C}_{σ} is defined to be the interior of $\widetilde{\text{Nef}}_{\sigma}$; the *extended Kähler cone* of \mathcal{X} , $\widetilde{C}_{\mathcal{X}}$, is defined to be the interior of the extended nef cone $\widetilde{\text{Nef}}_{\mathcal{X}}$.

Let \mathbb{K}_{σ} be the dual lattice of $\mathbb{K}_{\sigma}^{\vee}$; it can be viewed as an additive subgroup of $\mathbb{L}_{\mathbb{Q}}$:

$$\mathbb{K}_{\sigma} = \{\beta \in \mathbb{L}_{\mathbb{Q}} \mid \langle D, \beta \rangle \in \mathbb{Z} \ \forall D \in \mathbb{K}_{\sigma}^{\vee}\},$$

where $\langle -, - \rangle$ is the natural pairing between $\mathbb{L}_{\mathbb{Q}}^{\vee}$ and $\mathbb{L}_{\mathbb{Q}}$. Define

$$\mathbb{K} := \bigcup_{\sigma \in \Sigma(n)} \mathbb{K}_{\sigma}.$$

Then \mathbb{K} is a subset (which is not necessarily a subgroup) of $\mathbb{L}_{\mathbb{Q}}$, and $\mathbb{L} \subset \mathbb{K}$.

We define the *extended σ -Mori cone* $\widetilde{\text{NE}}_{\sigma} \subset \mathbb{L}_{\mathbb{R}}$ to be the dual cone of $\widetilde{\text{Nef}}_{\sigma} \subset \mathbb{L}_{\mathbb{R}}^{\vee}$:

$$\widetilde{\text{NE}}_{\sigma} = \{\beta \in \mathbb{L}_{\mathbb{R}} \mid \langle D, \beta \rangle \geq 0 \ \forall D \in \widetilde{\text{Nef}}_{\sigma}\}.$$

It is a k -dimensional cone in $\mathbb{L}_{\mathbb{R}}$. The *extended Mori cone* of the extended stacky fan $(\Sigma, b_1, \dots, b_r)$ is

$$\widetilde{\text{NE}}_{\mathcal{X}} := \bigcup_{\sigma \in \Sigma(n)} \widetilde{\text{NE}}_{\sigma}.$$

Finally, we define

$$\mathbb{K}_{\text{eff},\sigma} := \mathbb{K}_\sigma \cap \widetilde{\text{NE}}_\sigma, \quad \mathbb{K}_{\text{eff}} := \mathbb{K} \cap \widetilde{\text{NE}}(\mathcal{X}) = \bigcup_{\sigma \in \Sigma(n)} \mathbb{K}_{\text{eff},\sigma}.$$

Example 2.5. (1) $\mathcal{X} = \mathcal{X}_{1,1,1}$.

$$\begin{aligned} \mathbb{K}^\vee &\cong 3\mathbb{Z}, \quad \widetilde{\text{Nef}}_\mathcal{X} = \mathbb{R}_{\leq 0}; \\ \mathbb{K} &\cong \frac{1}{3}\mathbb{Z}, \quad \widetilde{\text{NE}}_\mathcal{X} = \mathbb{R}_{\leq 0}, \quad \mathbb{K}_{\text{eff}} = \frac{1}{3}\mathbb{Z}_{\leq 0}. \end{aligned}$$

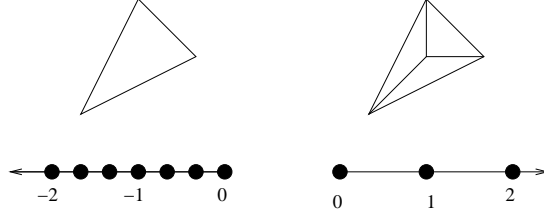


FIGURE 3. \mathbb{K}_{eff} of $\mathcal{X}_{1,1,1}$ and its crepant resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$

(2) $\mathcal{X} = \mathcal{X}_{1,2,0}$.

$$\begin{aligned} \mathbb{K}^\vee &\cong \mathbb{Z}(-2,1) \oplus \mathbb{Z}(1,-2), \quad \widetilde{\text{Nef}}_\mathcal{X} = \mathbb{R}_{\geq 0}(-2,1) + \mathbb{R}_{\geq 0}(1,-2); \\ \mathbb{K} &\cong \mathbb{Z}(-\frac{2}{3}, -\frac{1}{3}) \oplus \mathbb{Z}(-\frac{1}{3}, -\frac{2}{3}), \quad \widetilde{\text{NE}}_\mathcal{X} = \mathbb{R}_{\geq 0}(-\frac{2}{3}, -\frac{1}{3}) + \mathbb{R}_{\geq 0}(-\frac{1}{3}, -\frac{2}{3}), \\ \mathbb{K}_{\text{eff}} &= \mathbb{Z}_{\geq 0}(-\frac{2}{3}, -\frac{1}{3}) + \mathbb{Z}_{\geq 0}(-\frac{1}{3}, -\frac{2}{3}). \end{aligned}$$

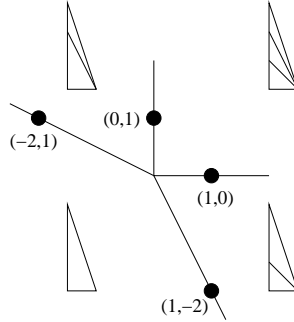


FIGURE 4. The secondary fan of the crepant resolution of $\mathcal{X}_{1,2,0}$

(3) $\mathcal{X} = \mathcal{X}_{0,0,0}$.

$$\begin{aligned} \mathbb{K}^\vee &\cong 3\mathbb{Z}, \quad \widetilde{\text{Nef}}_\mathcal{X} = \mathbb{R}_{\leq 0}; \\ \mathbb{K} &\cong \frac{1}{3}\mathbb{Z}, \quad \widetilde{\text{NE}}_\mathcal{X} = \mathbb{R}_{\leq 0}, \quad \mathbb{K}_{\text{eff}} = \frac{1}{3}\mathbb{Z}_{\leq 0}. \end{aligned}$$

Assumption 2.6. From now on, we make the following assumptions on the toric orbifold \mathcal{X} .

- (a) The coarse moduli space X_Σ of \mathcal{X} is semi-projective.
- (b) We may choose $b_{r'+1}, \dots, b_r$ such that $\hat{\rho} := D_1 + \dots + D_r$ is contained in the closure of the extended Kähler cone $\widetilde{C}_\mathcal{X}$.

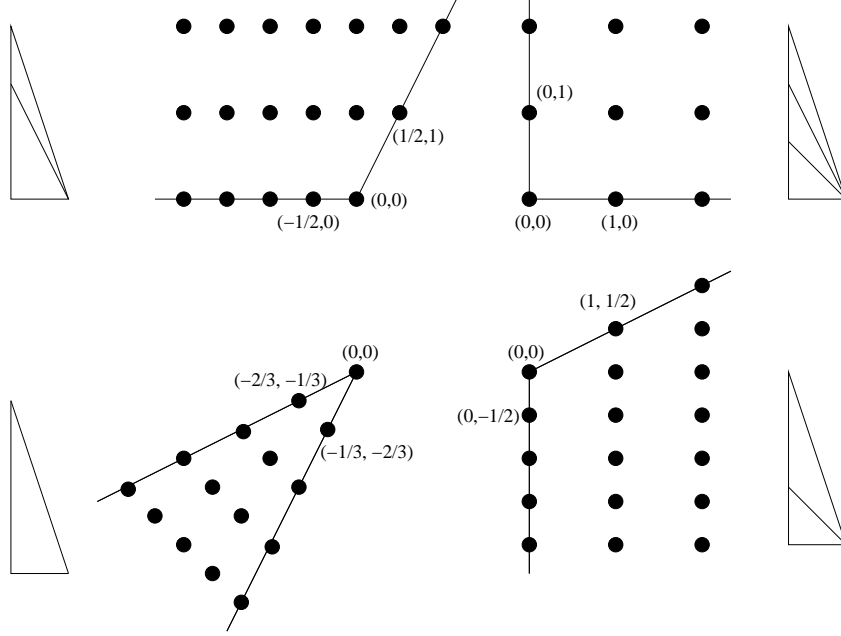


FIGURE 5. \mathbb{K}_{eff} of $\mathcal{X}_{1,2,0}$ and its (partial) crepant resolutions

- Remark 2.7.** (1) We make the above assumptions (a) and (b) so that the equivariant mirror theorem in [23] is applicable to \mathcal{X} .
- (2) By [24, Proposition 14.4.1], X_Σ is semi-projective if and only if $|\Sigma|$ is equal to the cone spanned by b_1, \dots, b_r . For example, $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ is a smooth toric Calabi-Yau 3-fold which is not semi-projective:

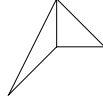


FIGURE 6. $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$

- (3) When \mathcal{X} is a Calabi-Yau smooth toric DM stack, Assumption (b) holds if its coarse moduli space X_Σ has a toric crepant resolution of singularities; see [37, Remark 3.4]. By [24, Proposition 11.4.19], any 3-dimensional Gorenstein toric variety X_Σ has a resolution of singularities $\phi : X_{\Sigma'} \rightarrow X_\Sigma$ such that ϕ is projective and crepant. So Assumption 2.6 (b) holds for any 3-dimensional Calabi-Yau smooth toric DM stacks.

2.6. Smooth toric DM stacks as symplectic quotients. Let $G_{\mathbb{R}} \cong U(1)^k$ be the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$. Then the Lie algebra of $G_{\mathbb{R}}$ is $\mathbb{L}_{\mathbb{R}}$. Let

$$\tilde{\mu} : \mathbb{C}^r \rightarrow \mathbb{L}_{\mathbb{R}}^{\vee} = \bigoplus_{a=1}^k \mathbb{R}e_a^{\vee}$$

be the moment map of the Hamiltonian $G_{\mathbb{R}}$ -action on \mathbb{C}^r , equipped with the Kähler form

$$\sqrt{-1} \sum_{i=1}^r dZ_i \wedge d\bar{Z}_i.$$

Then

$$\tilde{\mu}(Z_1, \dots, Z_r) = \sum_{i=1}^r \sum_{a=1}^k l_i^{(a)} |Z_i|^2 e_a.$$

If $\mathbf{r} = \sum_{a=1}^k r_a e_a^\vee$ is in the extended Kähler cone of \mathcal{X} , then

$$\mathcal{X} = [\tilde{\mu}^{-1}(\mathbf{r})/G_{\mathbb{R}}].$$

The real numbers r_1, \dots, r_k are extended Kähler parameters. Let $T_a = -r_a + \sqrt{-1}\theta_a$ be complexified extended Kähler parameters of \mathcal{X} .

2.7. The inertia stack and the Chen-Ruan orbifold cohomology. Given $\sigma \in \Sigma$, define

$$\text{Box}(\sigma) := \{v \in N : \bar{v} = \sum_{i \in I'_\sigma} c_i \bar{b}_i, \quad 0 \leq c_i < 1\}.$$

Then $N_{\text{tor}} \subset \text{Box}(\sigma) \subset N$. If $\tau \subset \sigma$ then $I'_\tau \subset I'_\sigma$, so $\text{Box}(\tau) \subset \text{Box}(\sigma)$.

Let $\sigma \in \Sigma(n)$ be a maximal cone in Σ . We have a short exact sequence of abelian groups

$$0 \rightarrow \mathbb{K}_\sigma/\mathbb{L} \rightarrow \mathbb{L}_{\mathbb{R}}/\mathbb{L} \rightarrow \mathbb{L}_{\mathbb{R}}/\mathbb{K}_\sigma \rightarrow 0,$$

which can be identified with the following short exact sequence of multiplicative abelian groups

$$1 \rightarrow G_\sigma \rightarrow G_{\mathbb{R}} \rightarrow (G/G_\sigma)_{\mathbb{R}} \rightarrow 0$$

where $G_{\mathbb{R}} \cong U(1)^k$ is the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$, and $(G/G_\sigma)_{\mathbb{R}} \cong U(1)^k$ is the maximal compact subgroup of $G_\sigma \cong (\mathbb{C}^*)^k$.

Given a real number x , we recall some standard notation: $\lfloor x \rfloor$ is the greatest integer less than or equal to x , $\lceil x \rceil$ is the least integer greater or equal to x , and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x . Define $v : \mathbb{K}_\sigma \rightarrow N$ by

$$v(\beta) = \sum_{i=1}^r \lceil \langle D_i, \beta \rangle \rceil b_i.$$

Then

$$\overline{v(\beta)} = \sum_{i \in I'_\sigma} \{-\langle D_i, \beta \rangle\} \bar{b}_i,$$

so $v(\beta) \in \text{Box}(\sigma)$. Indeed, v induces a bijection $\mathbb{K}_\sigma/\mathbb{L} \cong \text{Box}(\sigma)$.

For any $\tau \in \Sigma$ there exists $\sigma \in \Sigma(n)$ such that $\tau \subset \sigma$. The bijection $G_\sigma \rightarrow \text{Box}(\sigma)$ restricts to a bijection $G_\tau \rightarrow \text{Box}(\tau)$.

Define

$$\text{Box}(\Sigma) := \bigcup_{\sigma \in \Sigma} \text{Box}(\sigma) = \bigcup_{\sigma \in \Sigma(n)} \text{Box}(\sigma).$$

Then $N_{\text{tor}} \subset \text{Box}(\Sigma) \subset N$. There is a bijection $\mathbb{K}/\mathbb{L} \rightarrow \text{Box}(\Sigma)$.

Given $v \in \text{Box}(\sigma)$, where $\sigma \in \Sigma(d)$, define $c_i(v) \in [0, 1) \cap \mathbb{Q}$ by

$$\bar{v} = \sum_{i \in I'_\sigma} c_i(v) \bar{b}_i.$$

Suppose that $k \in G_\sigma$ corresponds to $v \in \text{Box}(\sigma)$ under the bijection $G_\sigma \cong \text{Box}(\sigma)$, then

$$\chi_i(k) = \begin{cases} 1, & i \in I_\sigma, \\ e^{2\pi\sqrt{-1}c_i(v)}, & i \in I'_\sigma. \end{cases}$$

Define

$$\text{age}(k) = \text{age}(v) = \sum_{i \notin I_\sigma} c_i(v).$$

Let $IU = \{(z, k) \in U_{\mathcal{A}} \times G \mid k \cdot z = z\}$, and let G acts on IU by $h \cdot (z, k) = (h \cdot z, k)$. The inertia stack \mathcal{IX} of \mathcal{X} is defined to be the quotient stack

$$\mathcal{IX} := [IU/G].$$

Note that $(z = (Z_1, \dots, Z_r), k) \in U$ if and only if

$$k \in \bigcup_{\sigma \in \Sigma} G_\sigma \text{ and } Z_i = 0 \text{ whenever } \chi_i(k) \neq 1.$$

So

$$IU = \bigcup_{v \in \text{Box}(\Sigma)} U_v,$$

where

$$U_v := \{(Z_1, \dots, Z_m) \in U_{\mathcal{A}} : Z_i = 0 \text{ if } c_i(v) \neq 0\}.$$

The connected components of \mathcal{IX} are

$$\{\mathcal{X}_v := [U_v/G] : v \in \text{Box}(\Sigma)\}.$$

The involution $IU \rightarrow IU$, $(z, k) \mapsto (z, k^{-1})$ induces involutions $\text{inv} : \mathcal{IX} \rightarrow \mathcal{IX}$ and $\text{inv} : \text{Box}(\Sigma) \rightarrow \text{Box}(\Sigma)$ such that $\text{inv}(\mathcal{X}_v) = \mathcal{X}_{\text{inv}(v)}$.

In the remainder of this subsection, we consider rational cohomology, and write $H^*(-)$ instead of $H^*(-; \mathbb{Q})$.

The Chen-Ruan orbifold cohomology [19] is defined to be

$$H_{\text{orb}}^*(\mathcal{X}) = \bigoplus_{v \in \text{Box}(\Sigma)} H^*(\mathcal{X}_v)[2\text{age}(v)].$$

Denote $\mathbf{1}_v$ to be the unit in $H^*(\mathcal{X}_v)$. Then $\mathbf{1}_v \in H_{\text{orb}}^{2\text{age}(v)}(\mathcal{X})$. In particular,

$$H_{\text{orb}}^0(\mathcal{X}) = \bigoplus_{v \in N_{\text{tor}}} \mathbb{Q}\mathbf{1}_v.$$

Suppose that \mathcal{X} is a *proper* toric DM stack. Then the orbifold Poincaré pairing on $H_{\text{orb}}^*(\mathcal{X})$ is defined as

$$(3) \quad (\alpha, \beta) := \int_{\mathcal{IX}} \alpha \cup \text{inv}^*(\beta),$$

We also have an equivariant pairing on $H_{\text{orb}, \mathbb{T}}^*(\mathcal{X})$:

$$(4) \quad (\alpha, \beta)_{\mathbb{T}} := \int_{\mathcal{IX}_{\mathbb{T}}} \alpha \cup \text{inv}^*(\beta),$$

where

$$\int_{\mathcal{IX}_{\mathbb{T}}} : H_{\text{orb}, \mathbb{T}}^*(\mathcal{X}) \rightarrow H_{\mathbb{T}}^*(\text{point}) = H^*(B\mathbb{T})$$

is the equivariant pushforward to a point. When \mathcal{X} is not proper, (3) is not defined, but we can still define via (4) an equivariant pairing $H_{\text{orb}, \mathbb{T}}^*(\mathcal{X}) \otimes H_{\text{orb}, \mathbb{T}}^*(\mathcal{X}) \rightarrow \mathcal{Q}_{\mathbb{T}}$, where $\mathcal{Q}_{\mathbb{T}}$ is the fractional field of the ring $H^*(B\mathbb{T})$.

Example 2.8. (1) $\mathcal{X} = \mathcal{X}_{1,1,1}$.

$$N = \mathbb{Z}^3, \quad \text{Box}(\Sigma) = \{(0, 0, 0), (0, 0, 1), (0, 0, 2)\};$$

$$H_{\text{orb}}^0(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,0)}, \quad H_{\text{orb}}^2(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,1)}, \quad H_{\text{orb}}^4(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,2)}.$$

(2) $\mathcal{X} = \mathcal{X}_{1,2,0}$.

$$N = \mathbb{Z}^3, \quad \text{Box}(\Sigma) = \{(0, 0, 0), (0, 2, 1), (0, 1, 1)\};$$

$$H_{\text{orb}}^0(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,0,0)}, \quad H_{\text{orb}}^2(\mathcal{X}) = \mathbb{Q}\mathbf{1}_{(0,2,1)} \oplus \mathbb{Q}\mathbf{1}_{(0,1,1)}.$$

(3) $\mathcal{X} = \mathcal{X}_{0,0,0}$.

$$N = \mathbb{Z}^3 \oplus \mathbb{Z}_3, \quad \text{Box}(\Sigma) = N_{\text{tor}} = \mathbb{Z}_3 = \{0, 1, 2\};$$

$$H_{\text{orb}}^0(\mathcal{X}) = \mathbb{Q}\mathbf{1}_0 \oplus \mathbb{Q}\mathbf{1}_1 \oplus \mathbb{Q}\mathbf{1}_2.$$

3. OPEN-CLOSED GW INVARIANTS

From now on, we consider 3-dimensional Calabi-Yau smooth toric DM stacks. Then $n = 3$, $\mathbb{T} \cong (\mathbb{C}^*)^3$ and $\mathbb{T}' \cong \{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid t_1 t_2 t_3 = 1\}$.

3.1. Aganagic-Vafa A-branes. In [5], Aganagic-Vafa introduced a class of Lagrangian submanifolds of semi-projective smooth toric Calabi-Yau 3-folds. It is straightforward to generalize this construction to 3-dimensional Calabi-Yau smooth toric DM stacks with semi-projective coarse moduli spaces.

Let $\mathcal{X} = [\tilde{\mu}^{-1}(\mathbf{r})/G_{\mathbb{R}}]$ be a 3-dimensional Calabi-Yau smooth toric DM stack, where $\mathbf{r} \in \tilde{C}(\mathcal{X}) \subset \mathbb{L}_{\mathbb{R}}^{\vee}$. $\tilde{\mu}^{-1}(\mathbf{r})$ is defined by

$$\sum_{i=1}^{k+3} l_i^{(a)} |X_i|^2 = r_a, \quad a = 1, \dots, k.$$

Write $X_i = \rho_i e^{\sqrt{-1}\phi_i}$, where $\rho_i = |X_i|$. An Aganagic-Vafa brane is a Lagrangian sub-orbifold of \mathcal{X} of the form

$$\mathcal{L} = [\tilde{L}/G_{\mathbb{R}}]$$

where

$$\tilde{L} = \{(X_1, \dots, X_r) \in \tilde{\mu}^{-1}(\mathbf{r}) : \sum_{i=1}^{k+3} \hat{l}_i^1 |X_i|^2 = c_1, \sum_{i=1}^{k+3} \hat{l}_i^2 |X_i|^2 = c_2, \sum_{i=1}^{k+3} \phi_i = \text{const}\}$$

for some $\hat{l}_i^\alpha \in \mathbb{Z}$, $\sum_{i=1}^{k+3} \hat{l}_i^\alpha = 0$, $\alpha = 1, 2$. An Aganagic-Vafa brane intersects a unique 1-dimensional orbit closure $\mathfrak{l}_\tau := \mathcal{V}(\tau)$, $\tau \in \Sigma(2)$. Let ℓ_τ be the coarse moduli of \mathfrak{l}_τ . We say \mathcal{L} is an inner (resp. outer) brane if $\ell_\tau \cong \mathbb{P}^1$ (resp. $\ell_\tau \cong \mathbb{C}$).

We next describe the first homology group of an Aganagic-Vafa brane.² We pick a fixed point \mathfrak{p}_σ in ℓ_τ , so that $(\tau, \sigma) \in F(\Sigma)$. Suppose that $I'_\sigma = \{i_1, i_2, i_3\}$. Then $\mathcal{X}_\sigma = [\mathbb{C}^3/G_\sigma]$. Let $(\tau_1, \sigma) = (\tau, \sigma)$, (τ_2, σ) and (τ_3, σ) be three flags in the toric graph in the counter-clockwise direction, such that

$$I'_{\tau_1} = \{i_2, i_3\}, \quad I'_{\tau_2} = \{i_3, i_1\}, \quad I'_{\tau_3} = \{i_1, i_2\}.$$

For $j = 1, 2, 3$, let $G_j = G_{\tau_j}$ and let $s_j = r(\tau_j, \sigma)$. Then there are exact short sequences of finite abelian groups:

$$1 \rightarrow G_j \rightarrow G_\sigma \xrightarrow{\chi_{i_j}} \mu_{s_j} \rightarrow 1, \quad j = 1, 2, 3.$$

By the Calabi-Yau condition, for any $k \in G_\sigma$,

$$\chi_{i_1}(k)\chi_{i_2}(k)\chi_{i_3}(k) = 1, \quad \text{age}(k) \in \mathbb{Z}.$$

There are canonical identifications

$$\begin{aligned} (5) \quad G_\sigma &\cong \{v \in N : \bar{v} = c_1 \bar{b}_{i_1} + c_2 \bar{b}_{i_2} + c_3 \bar{b}_{i_3} \in N_{\mathbb{Q}}, 0 \leq c_i < 1\}, \\ G_1 &\cong \{v \in N : \bar{v} = c_2 \bar{b}_{i_2} + c_3 \bar{b}_{i_3} \in N_{\mathbb{Q}}, 0 \leq c_i < 1\}, \\ G_2 &\cong \{v \in N : \bar{v} = c_1 \bar{b}_{i_1} + c_3 \bar{b}_{i_3} \in N_{\mathbb{Q}}, 0 \leq c_i < 1\}, \\ G_3 &\cong \{v \in N : \bar{v} = c_1 \bar{b}_{i_1} + c_2 \bar{b}_{i_2} \in N_{\mathbb{Q}}, 0 \leq c_i < 1\}. \end{aligned}$$

Then

$$\mathcal{L} = [\tilde{L}_\sigma/G_\sigma] \subset \mathcal{X}_\sigma = [\mathbb{C}^3/G_\sigma] \subset \mathcal{X}.$$

There is a G_σ -equivariant diffeomorphism $\tilde{L}_\sigma \cong S^1 \times \mathbb{C}$, where G_σ acts on $S^1 \times \mathbb{C}$ by

$$k \cdot (e^{\sqrt{-1}\theta}, u) = (\chi_{i_1}(k)e^{\sqrt{-1}\theta}, \chi_{i_2}(k)u).$$

We have a commutative diagram

$$\begin{array}{ccccc} G_\sigma & \longrightarrow & \tilde{L}_\sigma & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow & & \downarrow \\ \mu_{s_1} & \longrightarrow & S^1 & \longrightarrow & S^1/\mu_{s_1} \end{array}$$

where the columns are fibrations. So we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(\tilde{L}_\sigma) & \longrightarrow & \pi_1(\mathcal{L}) & \longrightarrow & \pi_0(G_\sigma) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(S^1/\mu_s) & \longrightarrow & \pi_0(\mu_{s_1}) \longrightarrow 0 \end{array}$$

²If the $G_{\mathbb{R}}$ -action on \tilde{L} is free, then \mathcal{L} is a smooth manifold diffeomorphic to $S^1 \times \mathbb{R}^2$, so $H_1(\mathcal{L}; \mathbb{Z}) = \mathbb{Z}$

where the rows are short exact sequences of abelian groups. The above diagram can be rewritten as:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(\mathcal{L}; \mathbb{Z}) & \xrightarrow{\pi_\sigma} & G_\sigma \longrightarrow 0 \\ & & \text{id} \downarrow & & \pi \downarrow & & \chi_{i_1} \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times s_1} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{s_1} \longrightarrow 0 \end{array}$$

The group homomorphism $\pi \times \pi_\sigma : H_1(\mathcal{L}; \mathbb{Z}) \rightarrow \mathbb{Z} \times G_\sigma$ is injective, and the image is

$$\{(d_0, k) \in \mathbb{Z} \times G_\sigma : \exp(2\pi\sqrt{-1}\frac{d_0}{s_1}) = \chi_{i_1}(k)\}.$$

Let $V_\tau = \{\sigma \in \Sigma(3) : (\tau, \sigma) \in F(\Sigma)\}$. If $V_\tau = \{\sigma\}$ then define

$$H_{\tau, \sigma} = \{(\pi(\gamma), \pi_\sigma(\gamma)) : \gamma \in H_1(\mathcal{L}; \mathbb{Z})\} \subset \mathbb{Z} \times G_\sigma$$

If $V_\tau = \{\sigma_+, \sigma_-\}$ then define

$$H_{\tau, \sigma_+, \sigma_-} = \{(\pi(\gamma), \pi_{\sigma_+}(\gamma), \pi_{\sigma_-}(\gamma)) : \gamma \in H_1(\mathcal{L}; \mathbb{Z})\} \subset \mathbb{Z} \times G_{\sigma_+} \times G_{\sigma_-}.$$

Let $\Sigma_c(2) = \{\tau \in \Sigma(2) : |V_\tau| = 2\}$. Then \mathcal{L} is an inner brane iff $\tau \in \Sigma_c(2)$. We may identify $H_1(\mathcal{L}; \mathbb{Z})$ with $H_{\tau, \sigma}$ (resp. $H_{\tau, \sigma_+, \sigma_-}$) if \mathcal{L} is an outer (resp. inner) brane. The identification gives a bijection $H_{\tau, \sigma_+, \sigma_-} \rightarrow H_{\tau, \sigma_-, \sigma_+}$, $(d_0, k^+, k^-) \mapsto (-d_0, k^-, k^+)$.

3.2. Moduli spaces of stable maps to $(\mathcal{X}, \mathcal{L})$. Stable maps to orbifolds with Lagrangian boundary conditions and their moduli spaces have been introduced in [20, Section 2].

Let $(\Sigma, x_1, \dots, x_n)$ be a prestable bordered orbifold Riemann surface with n interior marked points in the sense of [20, Section 2]. Then the coarse moduli space $(\bar{\Sigma}, \bar{x}_1, \dots, \bar{x}_n)$ is a prestable bordered Riemann surface with n interior marked points, defined in [39, Section 3.6] and [44, Section 3.2]. We define the topological type (g, h) of Σ to be the topological type of $\bar{\Sigma}$ (see [44, Section 3.2]).

Let $(\Sigma, \partial\Sigma)$ be a prestable bordered orbifold Riemann surface of type (g, h) , and let $\partial\Sigma = R_1 \cup \dots \cup R_h$ be union of connected component. Each connected component is a circle which contains no orbifold points. Let $u : (\Sigma, \partial\Sigma) \rightarrow (\mathcal{X}, \mathcal{L})$ be a (bordered) stable map in the sense of [20, Section 2]. The topological type of u is given by the degree $\beta' = f_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$, and

$$\begin{cases} (\mu_i, k_i) = f_*[R_i] \in H_1(\mathcal{L}; \mathbb{Z}) = H_{\tau, \sigma} \subset \mathbb{Z} \times G_\sigma, & \text{if } \ell_\tau = \mathbb{C}, \\ (\mu_i, k_i^+, k_i^-) = f_*[R_i] \in H_1(\mathcal{L}; \mathbb{Z}) = H_{\tau, \sigma_+, \sigma_-} \subset \mathbb{Z} \times G_{\sigma_+} \times G_{\sigma_-}, & \text{if } \ell_\tau = \mathbb{P}^1. \end{cases}$$

We call μ_i winding numbers and k_i, k_i^\pm twistings. Given $\beta' \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$ and

$$\vec{\mu} = \begin{cases} ((\mu_1, k_1), \dots, (\mu_h, k_h)) \in H_1(\mathcal{L}; \mathbb{Z})^h & \text{if } \ell_\tau = \mathbb{C}, \\ ((\mu_1, k_1^+, k_1^-), \dots, (\mu_h, k_h^+, k_h^-)) \in H_1(\mathcal{L}; \mathbb{Z})^h & \text{if } \ell_\tau = \mathbb{P}^1. \end{cases}$$

let $\overline{\mathcal{M}}_{(g, h), n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ be the moduli space of stable maps of type (g, h) , degree β' , winding numbers and twisting $\vec{\mu}$, with n interior marked points.

The tangent space \mathcal{T}_ξ^1 and the obstruction space \mathcal{T}_ξ^2 at

$$\xi = [u : ((\Sigma, x_1, \dots, x_n), \partial\Sigma) \rightarrow (\mathcal{X}, \mathcal{L})] \in \overline{\mathcal{M}}_{(g, h), n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$$

fit into the following exact sequence:

$$\begin{aligned} 0 & \rightarrow \text{Aut}((\Sigma, x_1, \dots, x_n), \partial\Sigma) \rightarrow H^0(\Sigma, \partial\Sigma, u^*T\mathcal{X}, (u|_{\partial\Sigma})^*T\mathcal{L}) \rightarrow \mathcal{T}_\xi^1 \\ & \rightarrow \text{Def}((\Sigma, x_1, \dots, x_n), \partial\Sigma) \rightarrow H^1(\Sigma, \partial\Sigma, u^*T\mathcal{X}, (u|_{\partial\Sigma})^*T\mathcal{L}) \rightarrow \mathcal{T}_\xi^2, \end{aligned}$$

where $\text{Aut}((\Sigma, x_1, \dots, x_n), \partial\Sigma)$ (resp. $\text{Def}((\Sigma, x_1, \dots, x_n), \partial\Sigma)$) is the space of infinitesimal automorphisms (resp. deformations) of the domain. When Σ is smooth,

$$\begin{aligned} \text{Aut}((\Sigma, x_1, \dots, x_n), \partial\Sigma) &= H^0(\Sigma, \partial\Sigma, T\Sigma(-\sum_{j=1}^n x_j), T\partial\Sigma), \\ \text{Def}((\Sigma, x_1, \dots, x_n), \partial\Sigma) &= H^1(\Sigma, \partial\Sigma, T\Sigma(-\sum_{j=1}^n x_j), T\partial\Sigma). \end{aligned}$$

There are evaluation maps (at interior marked points)

$$\text{ev}_j : \overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) \rightarrow \mathcal{IX}, \quad j = 1, \dots, n.$$

Given $\vec{v} = (v_1, \dots, v_n)$, where $v_1, \dots, v_n \in \text{Box}(\Sigma)$, define

$$\overline{\mathcal{M}}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) := \bigcap_{j=1}^n \text{ev}_j^{-1}(\mathcal{X}_{v_j}).$$

Then the virtual (real) dimension of $\overline{\mathcal{M}}_{g,\vec{v}}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ is

$$2 \sum_{j=1}^n (1 - \text{age}(v_j)).$$

where $\text{age}(v_j) \in \{0, 1, 2\}$.

3.3. Torus action and equivariant invariants. Let $\mathbb{T}'_{\mathbb{R}} \cong U(1)^2$ be the maximal compact subgroup of $\mathbb{T}' \cong (\mathbb{C}^*)^2$. Then the $\mathbb{T}'_{\mathbb{R}}$ -action on \mathcal{X} is holomorphic and preserves \mathcal{L} , so it acts on the moduli spaces $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$. Given $\gamma_1, \dots, \gamma_n \in H_{\mathbb{T}'_{\mathbb{R}}, \text{orb}}^*(\mathcal{X}; \mathbb{Q}) = H_{\mathbb{T}'_{\mathbb{R}}, \text{orb}}^*(\mathcal{X}, \mathbb{Q})$, we define

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta', \vec{\mu}}^{\mathcal{X}, \mathcal{L}, \mathbb{T}'_{\mathbb{R}}} := \int_{[F]^{\text{vir}}} \frac{\prod_{j=1}^n (\text{ev}_j^* \gamma_i)|_F}{e_{\mathbb{T}'_{\mathbb{R}}}(N_F^{\text{vir}})} \in \mathcal{Q}_{\mathbb{T}'}$$

where $F \subset \overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ is the $\mathbb{T}'_{\mathbb{R}}$ -fixed points set of the $\mathbb{T}'_{\mathbb{R}}$ -action on $\overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ and $\mathcal{Q}_{\mathbb{T}'_{\mathbb{R}}} \cong \mathbb{Q}(\mathbf{w}_1, \mathbf{w}_2)$ is the fractional field of $H_{\mathbb{T}'_{\mathbb{R}}}^*$ (point; $\mathbb{Q}) \cong \mathbb{Q}[\mathbf{w}_1, \mathbf{w}_2]$.

3.4. Disk factor as equivariant open GW invariants. Suppose that \mathcal{L} is an inner brane. Let $(d, k_+, k_-) \in H_{\tau, \sigma_+, \sigma_-} \subset \mathbb{Z} \times G_{\sigma_+} \times G_{\sigma_-}$. Let

$$s_1^{\pm} = r(\tau, \sigma_{\pm}).$$

Suppose that

$$I'_{\sigma_+} = \{i_1, i_2, i_3\}, \quad I'_{\sigma_-} = \{i_4, i_3, i_2\}.$$

Let

$$L_2 = \mathcal{O}_{\mathcal{X}}(\mathcal{D}_{i_2})|_{\mathfrak{l}_{\tau}}, \quad L_3 = \mathcal{O}_{\mathcal{X}}(\mathcal{D}_{i_3})|_{\mathfrak{l}_{\tau}}.$$

Then

$$N_{\mathfrak{l}_{\tau}/\mathcal{X}} = L_2 \oplus L_3$$

Let

$$\mathbf{w}_j = (c_1)_{\mathbb{T}'}(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{i_j}))|_{\mathfrak{p}_+}, \quad j = 1, 2, 3.$$

Then $\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = 0$.

$$\begin{aligned} c_1(T_{\mathfrak{p}_+} \mathfrak{l}_{\tau}) &= \mathbf{w}_1 = \frac{\mathbf{u}}{s_1^+}, \quad c_1(L_2)|_{\mathfrak{p}_+} = \mathbf{w}_2, \quad c_1(L_3)|_{\mathfrak{p}_+} = \mathbf{w}_3, \\ c_1(T_{\mathfrak{p}_-} \mathfrak{l}_{\tau}) &= -\frac{\mathbf{u}}{s_1^-}, \quad c_1(L_2)|_{\mathfrak{p}_+} = \mathbf{w}_2 - a_2 \mathbf{u}, \quad c_1(L_3)|_{\mathfrak{p}_-} = \mathbf{w}_3 - a_3 \mathbf{u}, \end{aligned}$$

where $\deg L_i = a_i \in \mathbb{Q}$.

The Lagrangian \mathcal{L} intersects \mathfrak{l}_{τ} along a circle, which divides \mathfrak{l}_{τ} into two orbi-disks $D_+ \cong [D/\mathbb{Z}_{s_1^+}]$ and $D_- \cong [D/\mathbb{Z}_{s_1^-}]$, where $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, and \mathfrak{p}_{\pm} is the unique \mathcal{T} fixed point in D_{\pm} . Let

$$b = [D_+] \in H_2(\mathcal{X}, \mathcal{L}), \quad \alpha \in [\mathfrak{l}_{\tau}] \in H_2(\mathcal{X}), \quad \alpha - b \in H_2(\mathcal{X}, \mathcal{L}).$$

Let $(d_0, k^+, k^-) \in H_{\tau, \sigma_+, \sigma_-}$, where $d_0 \neq 0$. Define

$$\overline{\mathcal{M}}(d_0, k^+, k^-) := \begin{cases} \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \mathcal{L} \mid d_0 b, (d_0, k^+, k^-)), & d_0 > 0, \\ \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \mathcal{L} \mid -d_0(\alpha - b), (d_0, k^+, k^-)), & d_0 < 0. \end{cases}$$

$$\text{virtual dimension of } \overline{\mathcal{M}}(d_0, k^+, k^-) = \begin{cases} 1 - \text{age}(k^+), & d_0 > 0, \\ 1 - \text{age}(k^-), & d_0 < 0. \end{cases}$$

Define

$$D(d_0, k^+, k^-) := \begin{cases} \langle 1_{k^+} \rangle_{0, d_0 b, (d_0, k^+, k^-)}^{\mathcal{X}, \mathcal{L}}, & d_0 > 0, \\ \langle 1_{k^-} \rangle_{0, -d_0(\alpha-b), (d_0, k^+, k^-)}^{\mathcal{X}, \mathcal{L}}, & d_0 < 0. \end{cases}$$

Then $D(d_0, k^+, k^-)$ is a rational function in $\mathbf{w}_1, \mathbf{w}_2$, homogeneous of degree $\text{age}(k_+) - 1$ (resp. $\text{age}(k_-) - 1$) if $d_0 > 0$ (resp. $d_0 < 0$).

Similarly, when \mathcal{L} is an outer brane, we define $D(d_0, k) \in \mathbb{Q}(\mathbf{w}_1, \mathbf{w}_2)$, where $d_0 > 0$ and $k \in G_\sigma$.

The disk factor is computed in [10] when G_σ is cyclic, and in [53, Section 3.3] for general G_σ . In our notation, the formula in [53, Section 3.3] says³

$$(6) \quad \begin{aligned} D(d_0, k) &= \left(\frac{s_1 \mathbf{w}_1}{d_0} \right)^{\text{age}(k)-1} \frac{s_1}{d_0 |G_\sigma|} \cdot \frac{\prod_{a=1}^{\frac{d_0}{s_1} + c_{i_2}(k) + c_{i_3}(k) - 1} \left(\frac{d_0 \mathbf{w}_2}{\mathbf{w}_1} + a - c_{i_2}(k) \right)}{\lfloor \frac{d_0}{s_1} \rfloor!} \\ &= \left(\frac{s_1 \mathbf{w}_1}{d_0} \right)^{\text{age}(k)-1} \frac{1}{d_0 |G_1|} \cdot \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1} \rfloor + \text{age}(k) - 1} \left(\frac{d_0 \mathbf{w}_2}{\mathbf{w}_1} + a - c_{i_2}(k) \right)}{\lfloor \frac{d_0}{s_1} \rfloor!} \end{aligned}$$

where $c_i(k) \in \mathbb{Q} \cap [0, 1)$ is defined as in Section 2.7.

3.5. Disk factor as equivariant relative GW invariants. The disk factors $D(d_0, k^+, k^-)$ and $D(d_0, k)$ are rational functions in $\mathbf{w}_1, \mathbf{w}_2$. In order to obtain a rational number, we specialize to a 1-dimensional subtorus of $\mathbb{T}'_{\mathbb{R}}$ determined by the framing of the Aganagic-Vafa A-brane \mathcal{L} . When \mathcal{X} is a smooth toric Calabi-Yau 3-fold, the framing is an integer. In this section, we clarify the framing of an Aganagic-Vafa A-brane \mathcal{L} in a 3-dimensional Calabi-Yau smooth toric DM stack. We then reinterpret the disk factors $D(d_0, k^+, k^-)$ and $D(d_0, k)$ as equivariant relative Gromov-Witten invariants, which gives a canonical choice of the sign of the disk factor.

Let \mathcal{L} be an inner brane. Then there exists $f^+, f^- \in \mathbb{Z}$ such that

$$a_2 = \deg L_2 = \frac{f^+}{s_1^+} - \frac{f^- + 1}{s_1^-}, \quad a_3 = \deg L_3 = \frac{f^-}{s_1^-} - \frac{f^+ + 1}{s_1^+}.$$

We call such a choice (f^+, f^-) a *framing* of the Aganagic-Vafa brane \mathcal{L} . Given a framing (f^+, f^-) of \mathcal{L} , we degenerate \mathfrak{l}_τ to a nodal curve with two irreducible components \mathfrak{l}_+ and \mathfrak{l}_- , such that the stabilizer of the node \mathfrak{p}_0 is G_τ . Over the coarse curves ℓ_τ and ℓ_\pm of \mathfrak{l}_τ and \mathfrak{l}_\pm :

- we degenerate L_2 on ℓ_τ to $L_2^+ = \mathcal{O}(\frac{f^+}{s_1^+})$ on ℓ_+ and $L_2^- = \mathcal{O}(-\frac{f^-+1}{s_1^-})$ on ℓ_- ;
- we degenerate L_3 on ℓ_τ to $L_3^+ = \mathcal{O}(-\frac{f^++1}{s_1^+})$ on ℓ_+ and $L_3^- = \mathcal{O}(\frac{f^-}{s_1^-})$ on ℓ_- .

$$(c_1)_{\mathbb{T}'}(L_2^\pm)_{\mathfrak{p}_0} = \mathbf{w}_2 - f^+ \mathbf{w}_1, = -(c_1)_{\mathbb{T}'}(L_3^\pm)_{\mathfrak{p}_0}$$

We compute the disk factor $D(d_0, k^+, k^-)$ by computing genus zero relative stable maps to of $(\mathfrak{l}_+, \mathfrak{p}_+)$ or $(\mathfrak{l}_-, \mathfrak{p}_-)$. More precisely, let $\mathcal{M} = \mathcal{M}_{0,1}(\mathfrak{l}_+, \mathfrak{p}_0, (d_0))$ be the moduli space of relative stable maps $u : (\mathcal{C}, x, y) \rightarrow \mathfrak{l}_+$ such that $u^* \mathfrak{p}_0 = d_0 y$, where (\mathcal{C}, x, y) is a genus 0 orbicurve with 2 marked points. Let $\pi : \mathcal{U} \rightarrow \mathcal{M}$ be the universal curve and let $u : \mathcal{U} \rightarrow \mathcal{T}$ be the universal map to the universal target \mathcal{T} . Let $\text{ev} : \mathcal{M} \rightarrow \mathcal{I}\mathfrak{l}_+$ be the evaluation map at the (stacky) point x . We define

$$\langle 1_{k^+} \rangle_{0, d_0 b, k^+, k^-}^{\mathcal{X}, \mathcal{L}} = \int_{[\mathcal{M}_{0,1}(\mathfrak{l}_+, \mathfrak{p}_0, (d_0))]^{\text{vir}}} \text{ev}^*(1_{k^+}) e_{\mathbb{T}'}(-R^\bullet \pi_*(u^* L_2^+ \oplus u^* L_3^+ \ominus u^* L_2^+ \otimes \mathcal{O}_R)).$$

Let $u : (\mathcal{C}, x, y) \rightarrow \mathfrak{l}_+$ be relative stable map which represents a point in \mathcal{M} . Suppose that u is fixed by the torus action. Recall that $c_i : G_\sigma \rightarrow [0, 1) \cap \mathbb{Q}$ is defined by $\chi_i(k) = \exp(2\pi\sqrt{-1}c_i(k))$. For $j = 1, 2, 3$, let $\epsilon_j = c_{i_j}(k^+)$. Then $\epsilon_1 = \langle \frac{d_0}{s_1^+} \rangle$.

³The disk function in [53, Section 3.3] and our disk factor are the same when $k \neq 0$. When $k = 0$, the disk function is $\langle \rangle_{\dots}^{\mathcal{X}, \mathcal{L}}$ (no insertion), while the disk factor is $\langle 1 \rangle_{\dots}^{\mathcal{X}, \mathcal{L}}$ (one insertion of 1), so there is an additional factor of $(\frac{s_1}{\mathbf{w}_1})^{\delta_{0,k}}$ in the disk function in [53, Section 3.3].

$$\begin{aligned}
\text{ch}_{\mathbb{T}'}(H^0(\mathcal{C}, u^* L_1^+)) &= \sum_{a=0}^{\lfloor \frac{d_0}{s_1^+} \rfloor} e^{a \frac{s_1^+ w_1}{d_0}} \\
\text{ch}_{\mathbb{T}'}(H^1(\mathcal{C}, u^* L_1^+)) &= 0 \\
\text{ch}_{\mathbb{T}'}(H^0(\mathcal{C}, u^* L_2^+ \otimes \mathcal{O}_y)) &= \delta_{\langle \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rangle, 0} e^{w_2 - f^+ w_1} \\
\text{ch}_{\mathbb{T}'}(H^1(\mathcal{C}, u^* L_2^+ \otimes \mathcal{O}_y)) &= 0
\end{aligned}$$

$$\begin{aligned}
\text{ch}_{\mathbb{T}'}(H^0(\mathcal{C}, u^* L_2^+)) &= \begin{cases} \sum_{a=-\lfloor \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rfloor}^0 e^{w_2 + (a - \epsilon_2) \frac{s_1^+ w_1}{d_0}}, & f^+ \geq 0, \\ 0, & f^+ < 0, \end{cases} \\
\text{ch}_{\mathbb{T}'}(H^1(\mathcal{C}, u^* L_2^+)) &= \begin{cases} 0, & f^+ \geq 0, \\ -\lfloor \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rfloor - 1 \\ \sum_{a=1} e^{w_2 + (a - \epsilon_2) \frac{s_1^+ w_1}{d_0}}, & f^+ < 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{ch}_{\mathbb{T}'}(H^0(\mathcal{C}, u^* L_3^+)) &= \begin{cases} \sum_{a=-\lfloor -\frac{(f^+ + 1)d_0}{s_1^+} - \epsilon_3 \rfloor}^0 e^{-w_1 - w_2 + (a - \epsilon_3) \frac{s_1^+ w_1}{d_0}}, & f^+ < 0, \\ 0, & f^+ \geq 0, \end{cases} \\
\text{ch}_{\mathbb{T}'}(H^1(\mathcal{C}, u^* L_3^+)) &= \begin{cases} 0, & f^+ < 0, \\ -\lfloor -\frac{(f^+ + 1)d_0}{s_1^+} + 1 - \epsilon_3 \rfloor \\ \sum_{a=1} e^{-w_1 - w_2 + (a - \epsilon_3) \frac{s_1^+ w_1}{d_0}}, & f^+ \geq 0. \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\sum_{a=-\lfloor -\frac{(f^+ + 1)d_0}{s_1^+} - \epsilon_3 \rfloor}^0 e^{-w_1 - w_2 + (a - \epsilon_3) \frac{s_1^+ w_1}{d_0}} &= \sum_{a=\frac{d_0}{s_1^+} + \epsilon_2 + \epsilon_3}^{-\lfloor \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rfloor - 1 + \delta_{\langle \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rangle, 0}} e^{-w_2 + (\epsilon_2 - a) \frac{s_1^+ w_1}{d_0}} \\
-\lfloor \frac{-(f^+ + 1)d_0}{s_1^+} + 1 - \epsilon_3 \rfloor \sum_{a=1} e^{-w_1 - w_2 + (a - \epsilon_3) \frac{s_1^+ w_1}{d_0}} &= \sum_{a=-\lfloor \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rfloor + \delta_{\langle \frac{f^+ d_0}{s_1^+} \rangle, \epsilon_2}}^{\frac{d_0}{s_1^+} + \epsilon_2 + \epsilon_3 - 1} e^{-w_2 + (\epsilon_2 - a) \frac{s_1^+ w_1}{d_0}}
\end{aligned}$$

$$\frac{d_0}{s_1^+} + \epsilon_2 + \epsilon_3 = \lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+)$$

We have

$$\begin{aligned}
e_{\mathbb{T}'}(B_1^m) &= 1 \\
e_{\mathbb{T}'}(B_2^m) &= \lfloor \frac{d_0}{s_1^+} \rfloor! \left(\frac{s_1^+ w_1}{d_0} \right)^{\lfloor \frac{d_0}{s_1^+} \rfloor} \\
\frac{e_{\mathbb{T}'}(B_5^m)}{e_{\mathbb{T}'}(B_4^m)} &= (-1)^{\lceil \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rceil + \lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} \prod_{a=1}^{\lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} (w_2 + (a - \epsilon_2) \frac{s_1^+ w_1}{d_0}). \\
|\text{Aut}(f)| &= d_0 |G_1| \\
D(d_0, k^+, k^-) &= \frac{1}{|\text{Aut}(f)|} \frac{e_{\mathbb{T}'}(B_1^m) e_{\mathbb{T}'}(B_5^m)}{e_{\mathbb{T}'}(B_2^m) e_{\mathbb{T}'}(B_4^m)} \\
&= \frac{1}{d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} (w_2 + (a - \epsilon_2) \frac{s_1^+ w_1}{d_0})}{\lfloor \frac{d_0}{s_1^+} \rfloor! \left(\frac{s_1^+ w_1}{d_0} \right)^{\lfloor \frac{d_0}{s_1^+} \rfloor}} \cdot (-1)^{\lceil \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rceil + \lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} \\
&= (-1)^{\lceil \frac{f^+ d_0}{s_1^+} - \epsilon_2 \rceil + \lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} \left(\frac{s_1^+ w_1}{d_0} \right)^{\text{age}(k^+) - 1} \cdot \frac{1}{d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} (\frac{d_0 w_2}{s_1^+ w_1} + a - \epsilon_2)}{\lfloor \frac{d_0}{s_1^+} \rfloor!} \\
&= -(-1)^{\frac{d_0}{s_1^+}(-f^+ - 1) - \epsilon_3 - \{\epsilon_2 - f^+ \epsilon_1\}} \left(\frac{s_1^+ w_1}{d_0} \right)^{\text{age}(k^+) - 1} \cdot \frac{1}{d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} (\frac{d_0 w_2}{s_1^+ w_1} + a - \epsilon_2)}{\lfloor \frac{d_0}{s_1^+} \rfloor!}
\end{aligned}$$

If $d_0 > 0$, define

$$\begin{aligned}
D(d_0, k^+, k^-; f^+, f^-) \\
= -(-1)^{\frac{d_0}{s_1^+}(-f^+ - 1) - \epsilon_3 - \{\epsilon_2 - f^+ \epsilon_1\}} \left(\frac{s_1^+ w_1}{d_0} \right)^{\text{age}(k^+) - 1} \cdot \frac{1}{d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} (\frac{f^+ d_0}{s_1^+} - c_{i_2}(k^+) + a)}{\lfloor \frac{d_0}{s_1^+} \rfloor!}.
\end{aligned}$$

Let

$$\epsilon'_1 = c_{i_1}(k^-) = \langle -\frac{d_0}{s_1^-} \rangle, \quad \epsilon'_2 = c_{i_2}(k^-), \quad \epsilon'_3 = c_{i_3}(k^-).$$

If $d_0 < 0$, define

$$\begin{aligned}
D(d_0, k^+, k^-; f^+, f^-) \\
= -(-1)^{\frac{d_0}{s_1^-}(f^- + 1) - \epsilon'_2 - \{\epsilon'_3 - f^- \epsilon'_1\}} \left(\frac{s_1^+ w_1}{d_0} \right)^{\text{age}(k^-) - 1} \cdot \frac{1}{-d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{-d_0}{s_1^-} \rfloor + \text{age}(k^-) - 1} (\frac{-f^- d_0}{s_1^-} - c_{i_3}(k^-) + a)}{\lfloor \frac{-d_0}{s_1^-} \rfloor!} \\
= -(-1)^{\frac{d_0}{s_1^-}(-f^- - 1) + \epsilon'_2 - \{\epsilon_2 - f^+ \epsilon_1\}} \left(\frac{s_1^+ w_1}{d_0} \right)^{\text{age}(k^-) - 1} \cdot \frac{1}{-d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{-d_0}{s_1^-} \rfloor + \text{age}(k^-) - 1} (\frac{-f^- d_0}{s_1^-} - c_{i_3}(k^-) + a)}{\lfloor \frac{-d_0}{s_1^-} \rfloor!}
\end{aligned}$$

where we use the identity

$$\{\epsilon'_3 - f^- \epsilon'_1\} = \{-(\epsilon_2 - f^+ \epsilon_1)\}.$$

If \mathcal{L} is an outer brane, then the framing of \mathcal{L} is an integer f . Define

$$D(d_0, k; f) = -(-1)^{\frac{d_0}{s_1}(-f - 1) - \epsilon_3 - \{\epsilon_2 - f \epsilon_1\}} \left(\frac{s_1 w_1}{d_0} \right)^{\text{age}(k) - 1} \cdot \frac{1}{d_0 |G_1|} \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1} \rfloor + \text{age}(k) - 1} (\frac{f d_0}{s_1} - c_{i_2}(k) + a)}{\lfloor \frac{d_0}{s_1} \rfloor!}.$$

3.6. Open-closed GW invariants and descendant GW invariants. For any torus fixed point \mathfrak{p}_σ of \mathcal{X} , where $\sigma \in \Sigma(3)$, we have

$$H_{\text{orb}}^*(\mathfrak{p}_\sigma) = \bigoplus_{k \in G_\sigma} \mathbb{Q} \mathbf{1}_k, \quad H_{\text{orb}, \mathbb{T}'}^*(\mathfrak{p}_\sigma) = \bigoplus_{k \in G_\sigma} \mathbb{Q}[u_1, u_2] \mathbf{1}_k.$$

The inclusion $\iota_\sigma : \mathfrak{p}_\sigma \hookrightarrow \mathcal{X}$ induces

$$\iota_{\sigma*} : H_{\text{orb}, \mathbb{T}'}^*(\mathfrak{p}_\sigma) = H_{\mathbb{T}'}^*(\mathcal{I}\mathfrak{p}_\sigma) \rightarrow H_{\text{orb}, \mathbb{T}'}^*(\mathcal{X}) = H_{\mathbb{T}'}^*(\mathcal{I}\mathcal{X}).$$

Define

$$\phi_{\sigma, k} = \iota_{\sigma*} \mathbf{1}_k \in H_{\text{orb}, \mathbb{T}'}^*(\mathcal{X}).$$

We first assume that \mathcal{L} is an inner brane, let \mathfrak{l}_τ be the unique 1-dimensional orbit closure which intersects \mathcal{L} , and let $V_\tau = \{\sigma_+, \sigma_-\}$. Let

$$\vec{\mu} = ((\mu_1, k_1^+, k_1^-), \dots, (\mu_h, k_h^+, k_h^-)),$$

where $(\mu_j, k_j^+, k_j^-) \in H_{\tau, \sigma_+, \sigma_-}$. Let $J_\pm = \{j \in \{1, \dots, h\} : \pm \mu_j > 0\}$. Then there exists $\beta \in H_2(\mathcal{X})$ such that

$$\beta' = \beta + \left(\sum_{j \in J_+} \mu_j \right) b + \sum_{j \in J_-} (-\mu_j) (\alpha - b).$$

Let $\langle k_j^+ \rangle$ be the cyclic subgroup generated by k_j^+ , and let r_j^+ be the cardinality of $\langle k_j^+ \rangle$. Similarly, let r_j^- be the cardinality of $\langle k_j^- \rangle$.

We have

$$\begin{aligned} \overline{\mathcal{M}}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})^{\mathbb{T}'} &= \{F_\Gamma \mid \Gamma \in G_{g,n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})\} \\ \overline{\mathcal{M}}_{g,n+h}(\mathcal{X}, \beta)^{\mathbb{T}'} &= \{F_{\hat{\Gamma}} \mid \hat{\Gamma} \in G_{g,n+h}(\mathcal{X}, \beta)\} \end{aligned}$$

In the remaining part of this subsection, we use the following abbreviations:

$$\begin{aligned} \mathcal{M} &= \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}), \quad \hat{\mathcal{M}} = \overline{\mathcal{M}}_{g,n+h}(\mathcal{X}, \beta), \\ \mathcal{M}_j &= \begin{cases} \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \mathcal{L} \mid \mu_j b, (\mu_j, k_j^+, k_j^-)), & j \in J_+ \\ \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \mathcal{L} \mid -\mu_j(\alpha - b), (\mu_j, k_j^+, k_j^-)), & j \in J_- \end{cases} \\ \mathcal{G} &= G_{g,n}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}), \quad \hat{\mathcal{G}} = G_{g,n+h}(\mathcal{X}, \beta) \\ \mathbf{x} &= (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_h). \end{aligned}$$

Given $u : (\Sigma, \mathbf{x}, \partial\Sigma) \rightarrow (\mathcal{X}, \mathcal{L})$ which represents a point $\xi \in \mathcal{M}^{\mathbb{T}'}$, we have

$$\Sigma = \mathcal{C} \cup \bigcup_{j=1}^h D_j,$$

where \mathcal{C} is an orbicurve of genus g , $x_1, \dots, x_n \in \mathcal{C}$, $D_j = \{[z \in \mathbb{C} \mid |z| \leq 1] / \mathbb{Z}_{r_j^\pm}\}$, Σ and D_j intersect at $y_j = B\mathbb{Z}_{r_j^\pm}$ if $j \in J_\pm$. Let $u_j = u|_{D_j}$ and $\hat{u} = u|_{\mathcal{C}}$. Then

- (1) For $j = 1, \dots, h$, $u_j : (D_j, \partial D_j) \rightarrow (\mathcal{X}, \mathcal{L})$ represents a point in $\mathcal{M}_j^{\mathbb{T}'}$.
- (2) $\hat{u} : (\mathcal{C}, \mathbf{x}, \mathbf{y}) \rightarrow \mathcal{X}$ represents a point $\hat{\xi} \in \hat{\mathcal{M}}^{\mathbb{T}'}$, and $\hat{u}(\mathbf{y}_j) = [\mathfrak{p}_\pm, (k_j^\pm)^{-1}] \in \mathcal{I}\mathfrak{p}_\pm \subset \mathcal{I}\mathcal{X}$ if $j \in J_\pm$.

Let $x_{n+j} = y_j$. Let F_Γ be the connected component of $\mathcal{M}^{\mathbb{T}'}$ associated to the decorated graph $\Gamma \in \mathcal{G}$, and let $F_{\hat{\Gamma}}$ be the connected component of $\hat{\mathcal{M}}^{\mathbb{T}'}$ associated to the decorated graph $\Gamma' \in \hat{\mathcal{G}}$. Then for any $\Gamma \in \mathcal{G}$ there exists $\hat{\Gamma} \in \hat{\mathcal{G}}$ such that

$$\text{ev}_{n+j}(F_{\hat{\Gamma}}) = (\mathfrak{p}_\pm, (k_j^\pm)^{-1}) \in \mathcal{I}\mathfrak{p}_\pm \subset \mathcal{I}\mathcal{X}$$

if $j \in J_\pm$, and F_Γ can be identified with $F_{\hat{\Gamma}}$ up to a finite morphism. More precisely,

$$\begin{aligned} [F_\Gamma]^{\text{vir}} &= \prod_{j \in J_+} \frac{|G_{\sigma_+}|}{r_j^+ |\text{Aut}(u_j)|} \prod_{j \in J_-} \frac{|G_{\sigma_-}|}{r_j^- |\text{Aut}(u_j)|} [F_{\hat{\Gamma}}]^{\text{vir}} \\ &= \prod_{j \in J_+} \frac{s_1^+}{r_j^+ \mu_j} \prod_{j \in J_-} \frac{s_1^-}{-r_j^- \mu_j} [F_{\hat{\Gamma}}]^{\text{vir}}. \end{aligned}$$

We have

$$\frac{1}{e_{\mathbb{T}'}(N_{\Gamma}^{\text{vir}})} = \frac{e_{\mathbb{T}'}(B_1^m)e_{\mathbb{T}'}(B_5^m)}{e_{\mathbb{T}'}(B_2^m)e_{\mathbb{T}'}(B_4^m)}, \quad \frac{1}{e_{\mathbb{T}'}(N_{\hat{\Gamma}}^{\text{vir}})} = \frac{e_{\mathbb{T}'}(\hat{B}_1^m)e_{\mathbb{T}'}(\hat{B}_5^m)}{e_{\mathbb{T}'}(\hat{B}_2^m)e_{\mathbb{T}'}(\hat{B}_4^m)},$$

where

$$\begin{aligned} e_{\mathbb{T}'}(B_1^m) &= e_{\mathbb{T}'}(\hat{B}_1^m), \\ e_{\mathbb{T}'}(B_4^m) &= e_{\mathbb{T}'}(\hat{B}_4^m) \prod_{j \in J_+} \left(\frac{s_1^+ \mathbf{w}_1}{r_j^+ \mu_j} - \frac{\bar{\psi}_j}{r_j^+} \right) \prod_{j \in J_-} \left(\frac{s_1^+ \mathbf{w}_1}{r_j^- \mu_j} - \frac{\bar{\psi}_j}{r_j^-} \right) \end{aligned}$$

For $k = 0, 1$ and $j = 1, \dots, h$, let

$$H^k(D_j) = H^k(D_j, \partial D_j, u_j^* T\mathcal{X}, (u_j|_{\partial D_j})^* T\mathcal{L}).$$

Then there is a long exact sequence

$$\begin{aligned} 0 \rightarrow B_2 \rightarrow \hat{B}_2 \oplus \bigoplus_{j=1}^h H^0(D_j) &\rightarrow \bigoplus_{j \in J_+} (T_{\mathfrak{p}_+} \mathcal{X})^{k_j^+} \oplus \bigoplus_{j \in J_-} (T_{\mathfrak{p}_-} \mathcal{X})^{k_j^-} \\ \rightarrow B_5 \rightarrow \hat{B}_5 \oplus \bigoplus_{j=1}^h H^1(D_j) &\rightarrow 0, \end{aligned}$$

where $(T_{\mathfrak{p}_{\pm}} \mathcal{X})^{k_j^{\pm}}$ denote the k_j^{\pm} -invariant part of $T_{\mathfrak{p}_{\pm}} \mathcal{X}$. Note that

$$(T_{\mathfrak{p}_{\pm}} \mathcal{X})^{k_j^{\pm}} = T_{(\mathfrak{p}_{\pm}, k_j^{\pm})} \mathcal{I}\mathcal{X} = T_{(\mathfrak{p}_{\pm}, k_j^{-1})} \mathcal{I}\mathcal{X}.$$

$$\frac{e_{\mathbb{T}'}(H^1(D_j)^m)}{e_{\mathbb{T}'}(H^0(D_j)^m)} = |\mu_j| |G_1| D(\mu_j, k^+, k^-; f^+, f^-)$$

Then

$$\begin{aligned} &\int_{[F_{\Gamma}]^{\text{vir}}} \frac{(\prod_{i=1}^n \text{ev}_i^* \gamma_i)|_{F_{\Gamma}}}{e_{\mathbb{T}'}(N_{\Gamma}^{\text{vir}})} \\ &= \prod_{j \in J_+} \frac{s_1^+}{r_j^+ \mu_j} \prod_{j \in J_-} \frac{s_1^-}{-r_j^- \mu_j} \prod_{j=1}^h (|\mu_j| |G_1| D(\mu_j, k_j^+, k_j^-; f^+, f^-)) \\ &\quad \cdot \int_{[F_{\Gamma}]^{\text{vir}}} \frac{(\prod_{i=1}^n \text{ev}_i^* \gamma_i \prod_{j \in J_+} \text{ev}_{n+j}^* \phi_{\sigma_+, (k_j^+)^{-1}} \prod_{j \in J_-} \text{ev}_{n+j}^* \phi_{\sigma_-, (k_j^-)^{-1}})|_{F_{\Gamma}}}{\prod_{j \in J_+} (\frac{s_1^+ \mathbf{w}_1}{r_j^+ \mu_j} - \frac{\bar{\psi}_{n+j}}{r_j^+}) \prod_{j \in J_-} (\frac{s_1^+ \mathbf{w}_1}{r_j^- \mu_j} - \frac{\bar{\psi}_{n+j}}{r_j^-}) e_{\mathbb{T}'}(N_{\hat{\Gamma}}^{\text{vir}})} \\ &= \prod_{j=1}^h D'(\mu_j, k_j^+, k_j^-; f^+, f^-) \\ &\quad \cdot \int_{[F_{\Gamma}]^{\text{vir}}} \frac{(\prod_{i=1}^n \text{ev}_i^* \gamma_i \prod_{j \in J_+} \text{ev}_{n+j}^* \phi_{\sigma_+, (k_j^+)^{-1}} \prod_{j \in J_-} \text{ev}_{n+j}^* \phi_{\sigma_-, (k_j^-)^{-1}})|_{F_{\Gamma}}}{\prod_{j=1}^h \frac{s_1^+ \mathbf{w}_1}{\mu_j} (\frac{s_1^+ \mathbf{w}_1}{\mu_j} - \bar{\psi}_{n+j}) e_{\mathbb{T}'}(N_{\hat{\Gamma}}^{\text{vir}})} \end{aligned}$$

where

$$D'(d_0, k^+, k^-; f^+, f^-) = \begin{cases} -(-1)^{\frac{d_0}{s_1^+}(-f^+-1)-c_{i_3}(k_+)-\{c_{i_2}(k_+)-f^+c_{i_1}(k_+)\}} \frac{s_1^+}{d_0} \left(\frac{s_1^+ \mathbf{w}_1}{d_0}\right)^{\text{age}(k^+)} \\ \cdot \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1^+} \rfloor + \text{age}(k^+) - 1} \left(\frac{f^+ d_0}{s_1^+} - c_{i_2}(k^+) + a\right)}{\lfloor \frac{d_0}{s_1^+} \rfloor!}, & d_0 > 0, \\ -(-1)^{\frac{d_0}{s_1^-}(-f^- - 1) + c_{i_2}(k_-) - \{c_{i_2}(k_-) - f^- c_{i_1}(k_-)\}} \frac{s_1^-}{-d_0} \left(\frac{s_1^+ \mathbf{w}_1}{d_0}\right)^{\text{age}(k^-)} \\ \cdot \frac{\prod_{a=1}^{\lfloor \frac{-d_0}{s_1^-} \rfloor + \text{age}(k^-) - 1} \left(\frac{-f^- d_0}{s_1^-} - c_{i_2}(k^-) + a\right)}{\lfloor \frac{-d_0}{s_1^-} \rfloor!}, & d_0 < 0. \end{cases}$$

Proposition 3.1 (framed inner brane). *Suppose that (\mathcal{L}, f^+, f^-) is a framed inner brane, and*

$$\vec{\mu} = ((\mu_1, k_1^+, k_1^-), \dots, (\mu_h, k_h^+, k_h^-)),$$

where $(\mu_j, k_j^+, k_j^-) \in H_{\tau, \sigma_+, \sigma_-}$. Let $J_{\pm} = \{j \in \{1, \dots, h\} : \pm \mu_j > 0\}$. Then

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta', \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f^+, f^-)} = \prod_{j=1}^h D'(\mu_j, k_j^+, k_j^-; f^+, f^-) \cdot \int_{[\overline{\mathcal{M}}_{g, n+h}(\mathcal{X}, \beta)]^{\text{vir}}} \frac{(\prod_{i=1}^n \text{ev}_i^* \gamma_i \prod_{j \in J_+} \text{ev}_{n+j}^* \phi_{\sigma_+, (k_j^+)^{-1}} \prod_{j \in J_-} \text{ev}_{n+j}^* \phi_{\sigma_-, (k_j^-)^{-1}})}{\prod_{j=1}^h \frac{s_1^+ \mathbf{w}_1}{\mu_j} \left(\frac{s_1^+ \mathbf{w}_1}{\mu_j} - \bar{\psi}_{n+j}\right)}$$

where

$$\beta \in H_2(\mathcal{X}), \quad \beta' = \beta + \left(\sum_{j \in J_+} \mu_j\right)b - \left(\sum_{j \in J_-} \mu_j\right)(\alpha - b) \in H_2(\mathcal{X}, \mathcal{L}).$$

Suppose that (\mathcal{L}, f) is a framed outer brane, and $(d_0, k) \in H_{\sigma, \tau} \subset \mathbb{Z}_{>0} \times G_{\sigma}$. Define

$$D'(d_0, k; f) = -(-1)^{\frac{d_0}{s_1}(-f-1)-c_{i_3}(k)-\{c_{i_2}(k)-f c_{i_1}(k)\}} \frac{s_1}{d_0} \left(\frac{s_1 \mathbf{w}_1}{d_0}\right)^{\text{age}(k)} \cdot \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1} \rfloor + \text{age}(k) - 1} \left(\frac{f d_0}{s_1} - c_{i_2}(k) + a\right)}{\lfloor \frac{d_0}{s_1} \rfloor!}.$$

Proposition 3.2 (framed outer brane). *Suppose that (\mathcal{L}, f) is a framed inner brane, and $\vec{\mu} = ((\mu_1, k_1), \dots, (\mu_h, k_h))$, where $(\mu_j, k_j) \in H_{\tau, \sigma}$. Then*

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta', \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)} = \prod_{j=1}^h D'(\mu_j, k_j; f) \cdot \int_{[\overline{\mathcal{M}}_{g, n+h}(\mathcal{X}, \beta)]^{\text{vir}}} \frac{(\prod_{i=1}^n \text{ev}_i^* \gamma_i \prod_{j=1}^h \text{ev}_{n+j}^* \phi_{\sigma, (k_j)^{-1}})}{\prod_{j=1}^h \frac{s_1 \mathbf{w}_1}{\mu_j} \left(\frac{s_1 \mathbf{w}_1}{\mu_j} - \bar{\psi}_{n+j}\right)}$$

where

$$\beta \in H_2(\mathcal{X}), \quad \beta' = \beta + \left(\sum_{j=1}^h \mu_j\right)b.$$

3.7. Generating functions of open-closed GW invariants. Introduce variables $\{X_j \mid j = 1, \dots, h\}$ and let

$$\tau_2 = \sum_{i=1}^m \tau_i u_i$$

where u_1, \dots, u_m form a basis of $H_{\text{orb}}^2(\mathcal{X}; \mathbb{Q})$. We choose \mathbb{T}' -equivariant lifting of τ_2 as follows: for each $u_i \in H_{\text{orb}}^2(\mathcal{X}; \mathbb{Q})$, we choose the unique \mathbb{T}' -equivariant lifting $u_i^{\mathbb{T}'} \in H_{\text{orb}, \mathbb{T}'}^2(\mathcal{X}; \mathbb{Q})$ such that $\iota_{\sigma}^* u_i^{\mathbb{T}'} = 0 \in H_{\text{orb}, \mathbb{T}'}^2(\mathfrak{p}_{\sigma}; \mathbb{Q})$, where $\iota_{\sigma}^* : H_{\text{orb}, \mathbb{T}'}^2(\mathcal{X}; \mathbb{Q}) \rightarrow H_{\text{orb}, \mathbb{T}'}^2(\mathfrak{p}_{\sigma}; \mathbb{Q})$ is induced by the inclusion map $\iota_{\sigma} : \mathfrak{p}_{\sigma} \rightarrow \mathcal{X}$.

If \mathcal{L} is an outer brane, define

$$(7) \quad F_{g,h}^{\mathcal{X},\mathcal{L}}(\tau_2, Q^b, X_1, \dots, X_h) = \sum_{\beta', n \geq 0} \sum_{(\mu_j, k_j) \in H_{\tau, \sigma}} \frac{\langle (\tau_2)^n \rangle_{g, \beta, (\mu_1, k_1), \dots, (\mu_h, k_h)}^{\mathcal{X}, \mathcal{L}}}{n!} \prod_{j=1}^h (Q^b X_j)^{\mu_j} (\mathbf{1}_{k_1} \otimes \dots \otimes \mathbf{1}_{k_h})$$

which is a function which takes values in $H_{\text{orb}}^*(\mathfrak{p}_\sigma; \mathbb{C})^{\otimes h}$, where

$$H_{\text{orb}}^*(\mathfrak{p}_\sigma; \mathbb{C}) = \bigoplus_{k \in G_\sigma} \mathbb{C} \mathbf{1}_k.$$

If \mathcal{L} is an inner brane, define

$$(8) \quad F_{g,h}^{\mathcal{X},\mathcal{L}}(\tau_2, Q^b, X_1, \dots, X_h) = \sum_{\beta', n \geq 0} \sum_{(\mu_j, k_j^+, k_j^-) \in H_{\tau, \sigma_+, \sigma_-}} \frac{\langle (\tau_2)^n \rangle_{g, \beta, (\mu_1, k_1^+, k_1^-), \dots, (\mu_h, k_h^+, k_h^-)}^{\mathcal{X}, \mathcal{L}}}{n!} \cdot \prod_{\substack{j \in \{1, \dots, h\} \\ \mu_j > 0}} (Q^b X_j)^{\mu_j} \prod_{\substack{j \in \{1, \dots, h\} \\ \mu_j < 0}} (Q^{b-\alpha} X_j)^{\mu_j} (\mathbf{1}_{k_1^+} \otimes \dots \otimes \mathbf{1}_{k_h^+})$$

which is a function which takes values in $H_{\text{orb}}^*(\mathfrak{p}_{\sigma_+} \mathbb{C})^{\otimes h}$, where

$$H_{\text{orb}}^*(\mathfrak{p}_{\sigma_+} \mathbb{C}) = \bigoplus_{k \in G_{\sigma_+}} \mathbb{C} \mathbf{1}_k.$$

3.8. The equivariant J -function and the disk potential. Let $\{u_i\}_{i=1}^N$ be a homogeneous basis of $H_{\mathbb{T}, \text{orb}}^*(\mathcal{X}; \mathbb{Q})$, and $\{u^i\}_{i=1}^N$ be its dual basis. Define

$$\tau = \sum_{i=1}^N \tau_i u_i = \tau_0 + \tau_2 + \tau_{>2}$$

where

$$\tau_0 \in H_{\mathbb{T}, \text{orb}}^0(\mathcal{X}; \mathbb{C}), \quad \tau_2 \in H_{\mathbb{T}, \text{orb}}^2(\mathcal{X}; \mathbb{C}), \quad \tau_{>2} \in H_{\mathbb{T}, \text{orb}}^{>2}(\mathcal{X}; \mathbb{C}).$$

The J -function [56, 22, 30] is a $H_{\mathbb{T}, \text{orb}}^*(\mathcal{X})$ -valued function:

$$J(\tau, z) := 1 + \sum_{\beta \geq 0, n \geq 0} \frac{1}{n!} \sum_{i=1}^N \langle 1, \tau^n, \frac{u_i}{z - \psi} \rangle_{0, \beta}^{\mathcal{X}} u^i.$$

Then

$$\iota_\sigma^* J(\tau, z)|_{\mathbf{w}_1 = f \mathbf{w}_2} = \sum_{k \in G_\sigma} J_{\sigma, k}^f(\tau, z) \mathbf{1}_k,$$

where

$$J_{\sigma, k}^f(\tau, z) = 1 + \sum_{\beta \geq 0, n \geq 0} \frac{1}{n!} \sum_{i=1}^N \langle 1, \tau^n, \frac{\phi_{\sigma, k^{-1}}}{z - \psi} \rangle_{0, \beta}^{\mathcal{X}}.$$

As a special case of Proposition 3.2,

$$\begin{aligned} \langle \gamma_1, \dots, \gamma_n \rangle_{0, \beta + d_0 b, (d_0, k)}^{\mathcal{X}, (\mathcal{L}, f)} &= D'(d_0, k; f) \int_{[\mathcal{M}_{0, n+1}(\mathcal{X}, \beta)]^{\text{vir}}} \frac{(\prod_{i=1}^n \text{ev}_i^* \gamma_i \cup \text{ev}_{n+1}^* \phi_{\sigma, k^{-1}})}{\frac{s_1 \mathbf{w}_1}{d_0} (\frac{s_1 \mathbf{w}_1}{d_0} - \bar{\psi}_{n+1})} \\ &= D'(d_0, k; f) \langle 1, \gamma_1, \dots, \gamma_n, \frac{\phi_{\sigma, k^{-1}}}{\frac{s_1 \mathbf{w}_1}{d_0} - \bar{\psi}} \rangle_{0, \beta}^{\mathcal{X}}; \end{aligned}$$

$$\begin{aligned} F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}(\tau_2, Q^b, X_1) &= \sum_{\beta, n \geq 0} \sum_{(d_0, k) \in H_{\tau, \sigma}} \frac{1}{n!} \langle (\tau_2)^n \rangle_{0, \beta + d_0 b, (d_0, k)}^{\mathcal{X}, (\mathcal{L}, f)} (Q^b X)^{d_0} \\ &= \sum_{(d_0, k) \in H_{\tau, \sigma}} (Q^b X)^{d_0} D'(d_0, k; f) J_{\sigma, k}^f(\tau_2, \frac{s_1 \mathbf{w}_1}{d_0}) \mathbf{1}_k. \end{aligned}$$

Proposition 3.3. *Let $X = Q^b X_1$. If (\mathcal{L}, f) is a framed outer brane, then*

$$F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\tau_2, Q^b, X_1) = \sum_{(d_0, k) \in H_{\tau, \sigma}} X^{d_0} D'(d_0, k; f) J_{\sigma, k}^f(\tau_2, \frac{s_1 w_1}{d_0}) \mathbf{1}_k$$

If (\mathcal{L}, f^+, f^-) is a framed inner brane, then

$$\begin{aligned} & F_{0,1}^{\mathcal{X},(\mathcal{L},f^+,f^-)}(\tau_2, Q^b, X_1) \\ &= \sum_{(d_0, k^+, k^-) \in H_{\tau, \sigma_+, \sigma_-}, d_0 > 0} X^{d_0} D'(d_0, k^+, k^-; f^+, f^-) J_{\sigma^+, k^+}^{f^+}(\tau_2, \frac{s_1^+ w_1}{d_0}) \mathbf{1}_{k^+} \\ &+ \sum_{(d_0, k^+, k^-) \in H_{\tau, \sigma_+, \sigma_-}, d_0 < 0} X^{d_0} Q^{-d_0 \alpha} D'(d_0, k^+, k^-; f^+, f^-) J_{\sigma^-, k^-}^{f^-}(\tau_2, \frac{s_1^+ w_1}{d_0}) \cdot \frac{s_1^+}{s_1} \mathbf{1}_{k^+} \end{aligned}$$

4. MIRROR SYMMETRY FOR THE DISK AMPLITUDES

4.1. The equivariant I -function and the equivariant mirror theorem. We choose a $p_1, \dots, p_k \in \mathbb{L}^\vee \cap \tilde{\mathcal{C}}_{\mathcal{X}}$ such that

- $\{p_1, \dots, p_k\}$ is a \mathbb{Q} -basis of $\mathbb{L}_{\mathbb{Q}}^\vee$.
- $\{\bar{p}_1, \dots, \bar{p}_{k'}\}$ is a \mathbb{Q} -basis of $H^2(\mathcal{X}; \mathbb{Q})$, and $\bar{p}_a = 0$ for $k' + 1 \leq a \leq k$.

Let q'_0, q_1, \dots, q_k be k formal variables, and define $q^\beta = q_1^{\langle p_1, \beta \rangle} \dots q_k^{\langle p_k, \beta \rangle}$ for $\beta \in \mathbb{K}$. The equivariant I -function is an $H_{\text{orb}, \tau}^*(\mathcal{X})$ -valued power series defined as follows [37]:

$$\begin{aligned} I(q'_0, q, z) &= e^{\frac{\log q'_0 + \sum_{a=1}^{k'} \bar{p}_a^T \log q_a}{z}} \sum_{\beta \in \mathbb{K}_{\text{eff}}} q^\beta \prod_{i=1}^{r'} \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^{\infty} (\bar{D}_i^T + (\langle D_i, \beta \rangle - m)z)}{\prod_{m=0}^{\infty} (\bar{D}_i^T + (\langle D_i, \beta \rangle - m)z)} \\ &\cdot \prod_{i=r'+1}^r \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^{\infty} (\langle D_i, \beta \rangle - m)z}{\prod_{m=0}^{\infty} (\langle D_i, \beta \rangle - m)z} \mathbf{1}_{v(\beta)} \end{aligned}$$

where $q^\beta = \prod_{a=1}^k q_a^{\langle p_a, \beta \rangle}$. Note that $\langle p_a, \beta \rangle \geq 0$ for $\beta \in \mathbb{K}_{\text{eff}}$. The equivariant I -function can be rewritten as

$$\begin{aligned} I(q'_0, q, z) &= e^{\frac{\log q'_0 + \sum_{a=1}^{k'} \bar{p}_a^T \log q_a}{z}} \sum_{\beta \in \mathbb{K}_{\text{eff}}} \frac{q^\beta}{z^{\langle \hat{\rho}, \beta \rangle + \text{age}(v(\beta))}} \prod_{i=1}^{r'} \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^{\infty} (\frac{\bar{D}_i^T}{z} + \langle D_i, \beta \rangle - m)}{\prod_{m=0}^{\infty} (\frac{\bar{D}_i^T}{z} + \langle D_i, \beta \rangle - m)} \\ &\cdot \prod_{i=r'+1}^r \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^{\infty} (\langle D_i, \beta \rangle - m)}{\prod_{m=0}^{\infty} (\langle D_i, \beta \rangle - m)} \mathbf{1}_{v(\beta)} \end{aligned}$$

where $\hat{\rho} = D_1 + \dots + D_r \in \tilde{\mathcal{C}}_{\mathcal{X}}$.

Suppose that \mathcal{X} is Calabi-Yau, so that $\text{age}(v)$ is an integer for any $v \in \text{Box}(\Sigma)$. Then

$$H_{\text{orb}, \tau}^{\leq 2}(\mathcal{X}) = H_{\text{orb}, \tau}^0(\mathcal{X}) \oplus H_{\text{orb}, \tau}^2(\mathcal{X}).$$

Let $\mathcal{Q} = \mathbb{Q}(u_1, u_2, u_3)$ be the fractional field of $H_{\tau}^*(\text{point}; \mathbb{Q}) = H_{\mathbb{T}}^*(\text{point}; \mathbb{Q})$.

$$\begin{aligned} H_{\text{orb}, \tau}^0(\mathcal{X}; \mathcal{Q}) &= \bigoplus_{v \in N_{\text{tor}}} \mathcal{Q} \mathbf{1}_v, \\ H_{\text{orb}, \tau}^2(\mathcal{X}; \mathcal{Q}) &= \bigoplus_{i=1}^{k'} \bigoplus_{v \in N_{\text{tor}}} \mathcal{Q} \bar{p}_a \mathbf{1}_v \oplus \bigoplus_{\substack{v \in \text{Box}(\Sigma) \\ \text{age}(v)=1}} \mathcal{Q} \mathbf{1}_v \end{aligned}$$

From now on, we further assume that $N_{\text{tor}} = 0$, or equivalently, \mathcal{X} has trivial generic stabilizer. In this case, we may assume that $\hat{\rho} = 0$, and

$$\{v \in \text{Box}(\Sigma) : \text{age}(v) = 1\} = \{b_{r'+1}, \dots, b_r\}.$$

Then

$$H_{\text{orb}, \mathcal{T}}^0(\mathcal{X}; \mathcal{Q}) = \mathcal{Q}\mathbf{1}, \quad H_{\text{orb}, \mathcal{T}}^2(\mathcal{X}; \mathcal{Q}) = \bigoplus_{a=1}^{k'} \mathcal{Q}\bar{p}_a \oplus \bigoplus_{a=r'+1}^r \mathcal{Q}\mathbf{1}_{b_i}.$$

We choose \bar{p}_a such that $\iota^* \bar{p}_a = 0$

$$\begin{aligned} I(q'_0, q, z) &= e^{\frac{\log q'_0 + \sum_{a=1}^{k'} \bar{p}_a^T \log q_a}{z}} \sum_{\beta \in \mathbb{K}_{\text{eff}}} \frac{q^\beta}{z^{\text{age}(v(\beta))}} \prod_{i=1}^{r'} \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^{\infty} (\frac{\bar{D}_i^T}{z} + \langle D_i, \beta \rangle - m)}{\prod_{m=0}^{\infty} (\frac{\bar{D}_i^T}{z} + \langle D_i, \beta \rangle - m)} \\ &\quad \cdot \prod_{i=r'+1}^r \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^{\infty} (\langle D_i, \beta \rangle - m)}{\prod_{m=0}^{\infty} (\langle D_i, \beta \rangle - m)} \mathbf{1}_{v(\beta)} \end{aligned}$$

For $i = 1, \dots, r$, we will define $\Omega_i \subset \mathbb{K}_{\text{eff}} - \{0\}$. and $A_i(q)$ supported on Ω_i . We observe that, if $\beta \in \mathbb{K}_{\text{eff}}$ and $v(\beta) = 0$ then $\langle D_i, \beta \rangle \in \mathbb{Z}$ for $i = 1, \dots, r$.

- For $i = 1, \dots, r'$, let

$$\Omega_i = \{\beta \in \mathbb{K}_{\text{eff}} : v(\beta) = 0, \langle D_i, \beta \rangle < 0 \text{ and } \langle D_j, \beta \rangle \geq 0 \text{ for } j \in \{1, \dots, r\} - \{i\}\}.$$

Then $\Omega_i \subset \{\beta \in \mathbb{K}_{\text{eff}} : v(\beta) = 0, \beta \neq 0\}$. We define

$$A_i(q) := \sum_{\beta \in \Omega_i} q^\beta \frac{(-1)^{-\langle D_i, \beta \rangle - 1} (-\langle D_i, \beta \rangle - 1)!}{\prod_{j \in \{1, \dots, r\} - \{i\}} \langle D_j, \beta \rangle!}.$$

- For $i = r' + 1, \dots, r$, let

$$\Omega_i := \{\beta \in \mathbb{K}_{\text{eff}} : v(\beta) = b_i, \langle D_j, \beta \rangle \notin \mathbb{Z}_{<0} \text{ for } j = 1, \dots, r\},$$

and define

$$A_i(q) = \sum_{\beta \in \Omega_i} q^\beta \prod_{j=1}^r \frac{\prod_{m=\lceil \langle D_j, \beta \rangle \rceil}^{\infty} (\langle D_j, \beta \rangle - m)}{\prod_{m=0}^{\infty} (\langle D_j, \beta \rangle - m)}.$$

Let σ be the smallest cone containing b_i . Then

$$b_i = \sum_{j \in I'_\sigma} c_j(b_i) b_j,$$

where $c_j(b_i) \in (0, 1)$ and $\sum_{j \in I'_\sigma} c_j(b_i) = 1$. There exists a unique $D_i^\vee \in \mathbb{L}_\mathbb{Q}$ such that

$$\langle D_j, D_i^\vee \rangle = \begin{cases} 1, & j = i, \\ -c_j(b_i), & j \in I'_\sigma, \\ 0, & j \in I_\sigma - \{i\}. \end{cases}$$

Then

$$A_i(q) = q^{D_i^\vee} + \text{higher order terms}$$

$$I(q'_0, q, z) = 1 + \frac{1}{z} (\log q'_0 \mathbf{1} + \sum_{a=1}^{k'} \log(q_a) \bar{p}_a^T + \sum_{i=1}^{r'} A_i(q) \bar{D}_i^T + \sum_{i=r'+1}^r A_i(q) \mathbf{1}_{b_i}) + o(z^{-1}).$$

For $i = 1, \dots, r'$,

$$\bar{D}_i^T = \sum_{a=1}^{k'} \bar{l}_i^{(a)} \bar{p}_a^T + \lambda_i$$

where $\lambda_i \in H^2(B\mathbb{T}; \mathbb{Q})$. Let $S_a(q) := \sum_{i=1}^{r'} \bar{l}_i^{(a)} A_i(q)$. Then

$$I(q'_0, q, z) = 1 + \frac{1}{z} ((\log q'_0 + \sum_{i=1}^{r'} \lambda_i A_i(q)) \mathbf{1} + \sum_{a=1}^{k'} (\log(q_a) + S_a(q)) \bar{p}_a^T + \sum_{i=r'+1}^r A_i(q) \mathbf{1}_{b_i}) + o(z^{-1}).$$

Recall that the equivariant small J -function for \mathcal{X} is

$$J(\boldsymbol{\tau}, z) = 1 + \sum_{\beta \geq 0, n \geq 0} \sum_{i=1}^N \frac{1}{n!} \langle 1, \boldsymbol{\tau}^n, \frac{u_i}{z - \psi} \rangle_{0, \beta}^{\mathcal{X}} u^i,$$

where $\{u_i\}_{i=1}^N$ is a $H^*(B\mathbb{T})$ -basis of $H_{\mathbb{T}}^*(\mathcal{X}; \mathbb{Q})$ and $\{u^i\}_{i=1}^N$ is the dual basis. Mirror theorem [30, 31, 41, 42] relates the small J -function

$$J(\boldsymbol{\tau}_0 + \boldsymbol{\tau}_2, z) = J(\tau_0 1 + \boldsymbol{\tau}_2, z) = e^{\frac{\tau_0}{z}} J(\boldsymbol{\tau}_2, z)$$

to the I -function up to a mirror transform. A mirror theorem for toric orbifolds will be proved in [23]. One version of that orbifold mirror theorem is stated in [37, Conjecture 4.3]. We cite its equivariant version as the following mirror theorem.

Conjecture 4.1. *If the toric orbifold \mathcal{X} satisfies Assumption 2.6, then*

$$e^{\frac{\tau_0(q'_0, q)}{z}} J(\boldsymbol{\tau}_2(q), z) = I(q'_0, q, z),$$

where the equivariant closed mirror map $(q'_0, q) \mapsto \tau_0(q'_0, q)1 + \boldsymbol{\tau}_2(q)$ is determined by the first-order term in the asymptotic expansion of the I -function

$$I(q'_0, q, z) = 1 + \frac{\tau_0(q'_0, q)1 + \boldsymbol{\tau}_2(q)}{z} + o(z^{-1}).$$

More explicitly, the equivariant closed mirror map is given by

$$\begin{aligned} \tau_0 &= \log(q'_0) + \sum_{i=1}^{r'} \lambda_i A_i(q), \\ \tau_a &= \begin{cases} \log(q_a) + S_a(q), & 1 \leq a \leq k', \\ A_{a-k'+r'}(q), & k' + 1 \leq a \leq k. \end{cases} \end{aligned}$$

Example 4.2. $\mathcal{X} = \mathcal{X}_{i,j,k}$, $k' = 0$.

$$\bar{D}_i \mathcal{T} = \begin{cases} \mathbf{w}_i, & i \in \{1, 2, 3\}, \\ 0, & i > 3. \end{cases}$$

(1) $\mathcal{X} = \mathcal{X}_{1,1,1}$, $k = 1$, $p_1 = -3$, $\mathbb{K}_{\text{eff}} = \frac{1}{3}\mathbb{Z}_{\leq 0}$.

$$A_1(q) = A_2(q) = A_3(q) = 0, \quad A_4(q) = \sum_{\alpha=0}^{\infty} \frac{(-1)^{3\alpha} q_1^{3\alpha+1}}{(3\alpha+1)!} \left(\frac{\Gamma(\alpha + \frac{1}{3})}{\Gamma(\frac{1}{3})} \right)^3.$$

$$\begin{aligned} I(q'_0, q_1, z) &= e^{\frac{\log q'_0}{z}} \left(\sum_{d \in \mathbb{Z}_{\geq 0}} \frac{q^d}{d! z^d} \prod_{i=1}^3 \prod_{m=\lceil -\frac{d}{3} \rceil}^{-1} (\mathbf{w}_i - (\frac{d}{3} + m)z) \mathbf{1}_{\omega^d} \right) \\ &= 1 + \frac{1}{z} \left((\log q'_0) \mathbf{1} + A_4(q) \mathbf{1}_{\omega} \right) + o(z^{-1}) \end{aligned}$$

Let $r = |q_1|$. The closed mirror map $\boldsymbol{\tau} = \tau_0 \mathbf{1} + \tau_1 \mathbf{1}_{\omega}$ is given by

$$\begin{aligned} \tau_0 &= \log q'_0, \\ \tau_1 &= A_4(q) = q_1 + O(r^4) \end{aligned}$$

(2) $\mathcal{X} = \mathcal{X}_{1,2,0}$, $k = 2$, $p_1 = (-2, 1)$, $p_2 = (1, -2)$, $\mathbb{K}_{\text{eff}} = \mathbb{Z}_{\geq 0}(-\frac{2}{3}, -\frac{1}{3}) \oplus \mathbb{Z}_{\geq 0}(-\frac{1}{3}, -\frac{2}{3})$,

$$A_1(q) = A_2(q) = A_3(q) = 0$$

$$A_4(q) = \sum_{\substack{d_1, d_2 \geq 0 \\ 2d_1 + d_2 \in 2 + 3\mathbb{Z}}} q_1^{d_1} q_2^{d_2} \frac{(-1)^{d_1 + d_2 - 1}}{d_1! d_2!} \frac{\Gamma(\frac{2d_1 + d_2}{3}) \Gamma(\frac{d_1 + 2d_2}{3})}{\Gamma(\frac{2}{3}) \Gamma(\frac{1}{3})}$$

$$A_5(q) = \sum_{\substack{d_1, d_2 \geq 0 \\ 2d_1 + d_2 = 1 + 3\mathbb{Z}}} q_1^{d_1} q_2^{d_2} \frac{(-1)^{d_1 + d_2 - 1}}{d_1! d_2!} \frac{\Gamma(\frac{2d_1 + d_2}{3}) \Gamma(\frac{d_1 + 2d_2}{3})}{\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})}$$

$$\begin{aligned}
& I(q'_0, q_1, q_2, z) \\
&= e^{\frac{\log q'_0}{z}} \sum_{d_1, d_2 \in \mathbb{Z}_{\geq 0}} \frac{q_1^{d_1} q_2^{d_2}}{d_1! d_2! z^{d_1+d_2}} \prod_{m=\lceil -\frac{2d_1+d_2}{3} \rceil}^{-1} (\mathbf{w}_3 - (\frac{2d_1+d_2}{3} + m)z) \\
&\quad \cdot \prod_{m=\lceil -\frac{d_1+2d_2}{3} \rceil}^{-1} (\mathbf{w}_2 - (\frac{d_1+2d_2}{3} + m)z) \mathbf{1}_{\omega \frac{d_1+2d_2}{3}} \\
&= 1 + \frac{1}{z} \left(\log q'_0 \mathbf{1} + A_4(q) \mathbf{1}_\omega + A_5(q) \mathbf{1}_{\omega^2} \right) + o(z^{-1})
\end{aligned}$$

Let $r = \sqrt{|q_1|^2 + |q_2|^2}$. The closed mirror map $\tau = \tau_0 \mathbf{1} + \tau_1 \mathbf{1}_\omega + \tau_2 \mathbf{1}_{\omega^2}$ is given by

$$\begin{aligned}
\tau_0 &= \log q'_0, \\
\tau_1 &= A_4(q) = q_1 + O(r^2), \\
\tau_2 &= A_5(q) = q_2 + O(r^2).
\end{aligned}$$

4.2. The pullback of the disk potential under the mirror map. By Proposition 3.3, if (\mathcal{L}, f) is a framed outer brane, then

$$F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\tau_2, Q^b, X_1) = \sum_{(d_0,k) \in H_{\tau,\sigma}} X^{d_0} D'(d_0, k; f) J_{\sigma,k}^f(\tau_2, \frac{s_1 \mathbf{w}_1}{d_0}) \mathbf{1}_k.$$

Let $F^{\mathcal{X},(\mathcal{L},f)}(q, X)$ be the pullback of $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\tau_2, Q^b, X_1)$ under the closed mirror map.

By Proposition 3.3, if (\mathcal{L}, f^+, f^-) is a framed inner brane, then

$$\begin{aligned}
& F_{0,1}^{\mathcal{X},(\mathcal{L},f^+,f^-)}(\tau_2, Q^b, X_1) \\
&= \sum_{(d_0,k^+,k^-) \in H_{\tau,\sigma_+,\sigma_-}, d_0 > 0} X^{d_0} D'(d_0, k^+, k^-; f^+, f^-) J_{\sigma^+,k^+}^{f^+}(\tau_2, \frac{s_1^+ \mathbf{w}_1}{d_0}) \mathbf{1}_{k^+} \\
&\quad + \sum_{(d_0,k^+,k^-) \in H_{\tau,\sigma_+,\sigma_-}, d_0 < 0} X^{d_0} Q^{-d_0 \alpha} D'(d_0, k^+, k^-; f^+, f^-) J_{\sigma^-,k^-}^{f^-}(\tau_2, \frac{s_1^+ \mathbf{w}_1}{d_0}) \cdot \frac{s_1^+}{s_1^-} \mathbf{1}_{k^+}.
\end{aligned}$$

Let $F^{\mathcal{X},(\mathcal{L},f^+,f^-)}(q, X)$ be the pullback of $F_{0,1}^{\mathcal{X},(\mathcal{L},f^+,f^-)}(\tau_2, Q^b, X_1)$ under the closed mirror map.

Given $\sigma \in \Sigma(3)$, $k \in G_\sigma$, and $f \in \mathbb{Z}$, define $I_{\sigma,k}^f(q, z)$ by

$$\iota_\sigma^* I(q, z)|_{\mathbf{w}_2 = f \mathbf{w}_1} = \sum_{k \in G_\sigma} I_{\sigma,k}^f(q, z) \mathbf{1}_k.$$

Since a toric Calabi-Yau orbifold satisfies the weak Fano condition, by the equivariant mirror theorem (Conjecture 4.1), we may write $F^{\mathcal{X},(\mathcal{L},f)}(q, X)$ in terms of $I_{\sigma,k}^f(q, z)$, and write $F^{\mathcal{X},(\mathcal{L},f^+,f^-)}(q, X)$ in terms of $I_{\sigma^+,k^+}^{f^+}(q, z)$ and $I_{\sigma^-,k^-}^{f^-}(q, z)$.

Theorem 4.3. *If (\mathcal{L}, f) is a framed outer brane, then*

$$(9) \quad F^{\mathcal{X},(\mathcal{L},f)}(q, X) = \sum_{(d_0,k) \in H_{\tau,\sigma}} X^{d_0} D'(d_0, k; f) e^{\frac{-d_0 \tau_0(q)}{s_1^+ \mathbf{w}_1}} I_{\sigma,k}^f(q, \frac{s_1 \mathbf{w}_1}{d_0}) \mathbf{1}_k$$

If (\mathcal{L}, f^+, f^-) is a framed inner brane, then

$$\begin{aligned}
& F^{\mathcal{X},(\mathcal{L},f^+,f^-)}(q, X) \\
&= \sum_{(d_0,k^+,k^-) \in H_{\tau,\sigma_+,\sigma_-}, d_0 > 0} X^{d_0} D'(d_0, k^+, k^-; f^+, f^-) e^{\frac{-d_0 \tau_0(q)}{s_1^+ \mathbf{w}_1}} I_{\sigma^+,k^+}^{f^+}(q, \frac{s_1^+ \mathbf{w}_1}{d_0}) \mathbf{1}_{k^+} \\
&\quad + \sum_{(d_0,k^+,k^-) \in H_{\tau,\sigma_+,\sigma_-}, d_0 < 0} X^{d_0} Q^{-d_0 \alpha} D'(d_0, k^+, k^-; f^+, f^-) e^{\frac{-d_0 \tau_0(q)}{s_1^+ \mathbf{w}_1}} I_{\sigma^-,k^-}^{f^-}(q, \frac{s_1^+ \mathbf{w}_1}{d_0}) \cdot \frac{s_1^+}{s_1^-} \mathbf{1}_{k^+}
\end{aligned}$$

4.2.1. *A framed outer brane* (\mathcal{L}, f) . Let (\mathcal{L}, f) be a framed outer brane. Let \mathfrak{l}_τ be the unique 1-dimensional orbit closure intersecting \mathcal{L} , and let \mathfrak{p}_σ be the unique torus fixed (stacky) point in \mathfrak{l}_τ . Recall that

$$\begin{aligned} I'_\sigma &= \{i \in \{1, \dots, r'\} : \rho_i \subset \sigma\} = \{i_1, i_2, i_3\}, \quad I_\sigma = \{1, \dots, r\} \setminus I'_\sigma, \\ I'_\tau &= \{i \in \{1, \dots, r'\} : \rho_i \subset \tau\} = \{i_2, i_3\}, \quad I_\tau = \{1, \dots, r\} \setminus I'_\tau, \\ \mathbb{K}_{\text{eff}, \sigma} &= \{\beta \in \mathbb{L}_\mathbb{Q} : \langle D_i, \beta \rangle \in \mathbb{Z}_{\geq 0} \text{ for } i \in I_\sigma\} \\ H_{\tau, \sigma} &= \{(d_0, k) \in \mathbb{Z} \times G_\sigma : \exp(2\pi\sqrt{-1}\frac{d_0}{s_1}) = \chi_{i_1}(k)\}. \end{aligned}$$

Let $\mathbf{b}_{\sigma, i} = \iota_\sigma^* \bar{D}_i^\mathcal{T} \in H^2(\mathfrak{p}_\sigma; \mathbb{Q}) = H^2(B\mathbb{T}; \mathbb{Q})$ for $1 \leq i \leq r'$, and let $\mathbf{b}_i = 0$ for $r' + 1 \leq i \leq r$. For $\beta \in \mathbb{K}_{\text{eff}, \sigma}$, define

$$(10) \quad I(\sigma, \beta) := \prod_{i=1}^r \frac{\prod_{m=\lceil \langle D_i, \beta \rangle \rceil}^\infty (\mathbf{b}_{\sigma, i} + (\langle D_i, \beta \rangle - m) \frac{s_1 \mathbf{w}_1}{d_0})}{\prod_{m=0}^\infty (\mathbf{b}_{\sigma, i} + (\langle D_i, \beta \rangle - m) \frac{s_1 \mathbf{w}_1}{d_0})}$$

Recall that $\iota_\sigma^* \bar{p}_a^\mathcal{T} = 0$, so

$$\iota_\sigma^* I(q, z)|_{z=\frac{s_1 \mathbf{w}_1}{d_0}} = \sum_{\beta \in \mathbb{K}_{\text{eff}, \sigma}} e^{\frac{d_0}{s_1 \mathbf{w}_1} \log q'_0} q^\beta I^f(\sigma, \beta) \mathbf{1}_{v(\beta)}$$

With the above notation, we can rewrite $F^{\mathcal{X}, (\mathcal{L}, f)}(q, X)$ as

$$F^{\mathcal{X}, (\mathcal{L}, f)}(q, X) = \sum_{(d_0, k) \in H_{\tau, \sigma}} \sum_{\beta \in \mathbb{K}_{\text{eff}, \sigma}, v(\beta)=k} x^{d_0} q^\beta D'(d_0, k, f) I^f(\sigma, \beta) \mathbf{1}_k$$

where $I^f(\sigma, \beta) = I(\sigma, \beta)|_{\mathbf{w}_2=f\mathbf{w}_1, \mathbf{w}_3=-(f+1)\mathbf{w}_1}$, and

$$x = X \exp\left(\frac{\log q'_0 - \tau_0(q)}{s_1 \mathbf{w}_1}\right)$$

is the B-brane moduli parameter. Recall that

$$\tau_0 + \sum_{a=1}^{k'} \tau_a \bar{p}_a^\mathcal{T} + \sum_{a=k'+1}^k \tau_a \mathbf{1}_{b_{a-k'+r'}} = \log q'_0 + \sum_{a=1}^{k'} \log q_a \bar{p}_a^\mathcal{T} + \sum_{i=1}^{r'} A_i(q) \bar{\mathcal{D}}_i^\mathcal{T} + \sum_{i=r'+1}^r A_i(q) \mathbf{1}_{b_i}.$$

We pull back the above identity under ι_σ^* , and recall that

$$\iota_\sigma^* \bar{p}_a^\mathcal{T} = 0, \quad \iota_\sigma^* \mathcal{D}_i^{\mathcal{T}'} \Big|_{\mathbf{w}_2=f\mathbf{w}_1, \mathbf{w}_3=(-f-1)\mathbf{w}_1} = l_i^{(0)} \mathbf{w}_1,$$

we get

$$\tau_0(q'_0, q) = \log q'_0 + \sum_{i=1}^{r'} l_i^{(0)} A_i(q) \mathbf{w}_1$$

So the open mirror map is given by

$$(11) \quad \log X = \log x + \frac{1}{s_1} \sum_{i=1}^{r'} l_i^{(0)} A_i(q).$$

We define $\{W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x)\}_{v \in \text{Box}(\sigma)}$ by

$$F^{\mathcal{X}, (\mathcal{L}, f)}(q, X) = \sum_{v \in \text{Box}(\sigma)} W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x) \mathbf{1}_v.$$

Then

$$(12) \quad W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x) = \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ \{\frac{d_0}{s_1}\} = c_{i_1}(v)}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}, \sigma} \\ v(\beta)=v}} x^{d_0} q^\beta D'(d_0, v, f) I^f(\sigma, \beta)$$

Following [40, 48], we define *extended charge vectors*

$$\{\tilde{l}_i^{(a)}\} = \begin{pmatrix} & & \{l_i^{(a)}\} & & 0 & 0 \\ \dots & 1 & \dots & f & \dots & -f-1 & \dots & 1 & -1 \end{pmatrix},$$

where $a = 0, \dots, k$, and the last row is an additional vector $\tilde{l}^{(0)}$ containing the phase and the framing of the A-brane \mathcal{L} . Given $\tilde{\beta} = (d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, define the extended or open sector pairing to be

$$\langle D_i, \tilde{\beta} \rangle = \frac{d_0}{s_1} \tilde{l}_i^{(0)} + \langle D_i, \beta \rangle.$$

Notice in this particular notation we allow $i = r + 1, r + 2$ since the corresponding $\tilde{l}_i^{(a)}$ exist, although D_i are not actual divisors.

Recall that $\{D_i : i \in I_\sigma\}$ is a \mathbb{Q} -basis of $\mathbb{L}_\mathbb{Q}^\vee \cong \mathbb{Q}^k$ and a \mathbb{Z} -basis of $\mathbb{K}_\sigma \cong \mathbb{Z}^k$. Let $\{p_a\}_{a=1, \dots, k} = \{D_i\}_{i \in I_\sigma}$, and let $\{e_a\}_{a=1, \dots, k}$ be the dual \mathbb{Q} -basis of $\mathbb{L}_\mathbb{Q}$, so that $\langle p_a, e_b \rangle = \delta_{ab}$. Then $\{e_a\}_{a=1, \dots, k}$ is a \mathbb{Z} -basis of $\mathbb{K}_\sigma \cong \mathbb{Z}^k$, and

$$\mathbb{K}_{\text{eff}, \sigma} = \sum_{a=0}^k \mathbb{Z}_{\geq 0} e_a.$$

Given any $(d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, define

$$q^{\tilde{\beta}} = x^{d_0} q^\beta = x^{d_0} \prod_{a=1}^k q_a^{\langle p_a, \beta \rangle}.$$

With the above notation, we have:

Theorem 4.4.

$$(13) \quad W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x) = \sum_{\substack{\tilde{\beta} = (d_0, \beta) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \\ v(\beta) = v}} q^{\tilde{\beta}} A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)}.$$

where

$$\begin{aligned} \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) &= \{\tilde{\beta} = (d_0, \beta) \in \mathbb{Z}_{>0} \times \mathbb{K}_{\text{eff}, \sigma} : \langle D_{i_1}, \tilde{\beta} \rangle \in \mathbb{Z}_{\geq 0}\} \\ A_{\tilde{\beta} = (d_0, \beta)}^{\mathcal{X}, (\mathcal{L}, f)} &= \frac{-(-1)^{\frac{d_0}{s_1}(-f-1) - \{c_{i_2}(v) - f c_{i_1}(v)\} + \langle D_{i_3}, \beta \rangle}}{\frac{d_0}{s_1} \prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} \cdot \frac{\Gamma(-\langle D_{i_3}, \tilde{\beta} \rangle)}{\Gamma(\langle D_{i_2}, \tilde{\beta} \rangle + 1)} \end{aligned}$$

Proof. To simplify the calculations, we introduce the symbol for any two numbers a and b with $a - b \in \mathbb{Z}$.

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\prod_{i=0}^{\infty} (a - i)}{\prod_{i=0}^{\infty} (b - i)}.$$

It has the following properties

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}^{-1}, \quad \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \begin{bmatrix} a \\ b \end{bmatrix} = (-1)^{a-b} \begin{bmatrix} -b-1 \\ -a-1 \end{bmatrix}.$$

Let $\tilde{\beta} = (d_0, \beta)$, and let $\epsilon_j = c_{i_j}(v)$ for $j = 1, 2, 3$. By (12),

$$W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x) = \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ \{\frac{d_0}{s_1}\} = \epsilon_1}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}, \sigma} \\ v(\beta) = v}} x^{d_0} q^\beta D'(d_0, v; f) I^f(\sigma, \beta),$$

where

$$\begin{aligned} D'(d_0, v; f) &= -(-1)^{\frac{d_0}{s}(-f-1) - \epsilon_3 - \{\epsilon_2 - f \epsilon_1\}} \frac{s_1}{d_0} \left(\frac{s_1 w_1}{d_0} \right)^{\epsilon_1 + \epsilon_2 + \epsilon_3} \frac{\prod_{a=1}^{\lfloor \frac{d_0}{s_1} \rfloor + \epsilon_1 + \epsilon_2 + \epsilon_3 - 1} (\frac{d_0}{s_1} f + (a - \epsilon_2))}{\lfloor \frac{d_0}{s_1} \rfloor!} \\ &= (-1)^{\frac{d_0}{s}(-f-1) - \epsilon_3 - \{\epsilon_2 - f \epsilon_1\}} \frac{s_1}{d_0} \left(\frac{s_1 w_1}{d_0} \right)^{\epsilon_1 + \epsilon_2 + \epsilon_3} \begin{bmatrix} 0 \\ \frac{d_0}{s_1} - \epsilon_1 \end{bmatrix} \left[\frac{(f+1)d_0}{s_1 f d_0} + \epsilon_3 - 1 \right] \end{aligned}$$

Given any $\beta \in \mathbb{K}_{\text{eff}, \sigma}$, we have $\lceil \langle D_{i_j}, \beta \rangle \rceil - \epsilon_j = \langle D_{i_j}, \beta \rangle$ for $j = 1, 2, 3$, and $\lceil \langle D_i, \beta \rangle \rceil = \langle D_i, \beta \rangle$ for $i \in I_\sigma$. By straightforward calculation,

$$\begin{aligned}
& I^f(\sigma, \beta) \\
&= \left(\frac{s_1 w_1}{d} \right)^{-\epsilon_1 - \epsilon_2 - \epsilon_3} \prod_{i \in I_\sigma} \frac{\prod_{a=-\infty}^0 a}{\prod_{a=-\infty}^{\lceil \langle D_i, \beta \rangle \rceil} a} \cdot \frac{\prod_{a=-\infty}^0 \left(\frac{d_0}{s_1} - \epsilon_1 + a \right)}{\prod_{a=-\infty}^{\lceil \langle D_{i_1}, \beta \rangle \rceil} \left(\frac{d_0}{s_1} - \epsilon_1 + a \right)} \\
&\quad \cdot \frac{\prod_{a=-\infty}^0 \left(f \frac{d_0}{s_1} - \epsilon_2 + a \right)}{\prod_{a=-\infty}^{\lceil \langle D_{i_2}, \beta \rangle \rceil} \left(f \frac{d_0}{s_1} - \epsilon_2 + a \right)} \cdot \frac{\prod_{a=-\infty}^0 \left((-1-f) \frac{d_0}{s_1} - \epsilon_3 + a \right)}{\prod_{a=-\infty}^{\lceil \langle D_{i_3}, \beta \rangle \rceil} \left((-1-f) \frac{d_0}{s_1} - \epsilon_3 + a \right)} \\
&= \left(\frac{s_1 w_1}{d_0} \right)^{-\epsilon_1 - \epsilon_2 - \epsilon_3} \left[\frac{\frac{d_0}{s_1} - \epsilon_1}{\frac{d_0}{s_1} + \langle D_{i_1}, \beta \rangle} \right] \left[\frac{\frac{f d_0}{s_1} - \epsilon_2}{\frac{f d_0}{s_1} + \langle D_{i_2}, \beta \rangle} \right] \left[\frac{\frac{(-f-1)d_0}{s_1} - \epsilon_3}{\frac{(-f-1)d_0}{s_1} + \langle D_{i_3}, \beta \rangle} \right] \prod_{i \in I_\sigma} \left[\frac{0}{\langle D_i, \beta \rangle} \right]
\end{aligned}$$

$$W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x) = \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ \{\frac{d_0}{s_1}\} = \epsilon_1}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}, \sigma} \\ v(\beta) = v}} x^{\frac{d_0}{s_1}} q^\beta A_{(d, \beta)}^{\mathcal{X}, (\mathcal{L}, f)}$$

where

$$A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)} = -(-1)^{\frac{d_0}{s_1}(-f-1) - \{\epsilon_2 - f\epsilon_1\} + \langle D_{i_3}, \beta \rangle} \frac{s_1}{d_0} \left[\frac{-\langle D_{i_3}, \tilde{\beta} \rangle - 1}{\langle D_{i_2}, \tilde{\beta} \rangle} \right] \prod_{i \in I_\tau} \left[\frac{0}{\langle D_i, \tilde{\beta} \rangle} \right].$$

Note that $A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)} = 0$ if $\langle D_i, \tilde{\beta} \rangle < 0$ for some $i \in I_\tau$, so

$$W_v^{\mathcal{X}, (\mathcal{L}, f)}(q, x) = \sum_{\substack{\tilde{\beta} = (d_0, \beta) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \\ v(\beta) = v}} q^{\tilde{\beta}} A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)},$$

where

$$A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)} = \frac{-(-1)^{\frac{d_0}{s_1}(-f-1) - \{c_{i_2}(v) - f c_{i_1}(v)\} + \langle D_{i_3}, \beta \rangle}}{\frac{d_0}{s_1} \prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} \cdot \frac{\Gamma(-\langle D_{i_3}, \tilde{\beta} \rangle)}{\Gamma(\langle D_{i_2}, \tilde{\beta} \rangle + 1)}$$

□

Example 4.5. $\mathcal{X} = \mathcal{X}_{i,j,k}$, the open mirror map is given by $x = X$.

(1) $\mathcal{X} = \mathcal{X}_{1,1,1}$, $(i_1, i_2, i_3) = (1, 2, 3)$, $s_1 = 3$. The extended charge vectors are

$$\begin{pmatrix} 1 & 1 & 1 & -3 & 0 & 0 \\ 1 & f & -f-1 & 0 & 1 & -1 \end{pmatrix}$$

$$F^{\mathcal{X}, (\mathcal{L}, f)}(q_1, X) = \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ d_1 \in \mathbb{Z}_{\geq 0} \\ d_0 - d_1 \in 3\mathbb{Z}}} x^{d_0} q_1^{d_1} \frac{-(-1)^{\lfloor \frac{d_1 - f d_0}{3} \rfloor - \frac{d_0 + 2d_1}{3}}}{\frac{d_0}{3} \cdot d_1! \left(\frac{d_0 - d_1}{3} \right)!} \cdot \frac{\Gamma\left(\frac{(f+1)d_0 + d_1}{3}\right)}{\Gamma\left(\frac{f d_0 - d_1}{3} + 1\right)} \mathbf{1}_{\omega^{d_1}}$$

(2) $\mathcal{X} = \mathcal{X}_{0,1,2}$, $(i_1, i_2, i_3) = (1, 2, 3)$, $s_1 = 1$. The extended charge vectors are

$$\begin{pmatrix} 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 1 & f & -f-1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned}
& F^{\mathcal{X}, (\mathcal{L}, f)}(q_1, q_2, X) \\
&= \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ d_1, d_2 \in \mathbb{Z}_{\geq 0}}} x^{d_0} q_1^{d_1} q_2^{d_2} \frac{-(-1)^{(-f-1)d_0 + \lfloor \frac{-2d_1 - d_2}{3} \rfloor}}{d_0 \cdot d_0! d_1! d_2!} \cdot \frac{\Gamma\left((f+1)d_0 + \frac{2d_1 + d_2}{3}\right)}{\Gamma\left(f d_0 - \frac{d_1 + 2d_2}{3} + 1\right)} \mathbf{1}_{\omega^{d_1 - d_2}}
\end{aligned}$$

(3) $\mathcal{X} = \mathcal{X}_{0,1,2}$, $(i_1, i_2, i_3) = (2, 3, 1)$. $s_1 = 3$. The extended charge vectors are

$$\begin{pmatrix} 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ -f-1 & 1 & f & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} & F^{\mathcal{X},(\mathcal{L},f)}(q_1, q_2, X) \\ &= \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ d_1, d_2 \in \mathbb{Z}_{>0} \\ d_0 - d_1 - 2d_2 \in 3\mathbb{Z}}} x^{d_0} q_1^{d_1} q_2^{d_2} \frac{-(-1)^{\lfloor \frac{(-f-1)d_0}{3} \rfloor}}{\frac{d_0}{3} \cdot (\frac{d_0 - d_1 - 2d_2}{3})! \cdot d_1! d_2!} \cdot \frac{\Gamma(\frac{(f+1)d_0}{3})}{\Gamma(\frac{fd_0 - 2d_1 - d_2}{3} + 1)} \mathbf{1}_{\omega^{d_1 - d_2}} \end{aligned}$$

(4) $\mathcal{X} = \mathcal{X}_{0,1,2}$, $(i_1, i_2, i_3) = (3, 1, 2)$. $s_1 = 3$. The extended charge vectors are

$$\begin{pmatrix} 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ f & -f-1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} & F^{\mathcal{X},(\mathcal{L},f)}(q_1, q_2, X) \\ &= \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ d_1, d_2 \in \mathbb{Z}_{>0} \\ d_0 - 2d_1 - d_2 \in 3\mathbb{Z}}} x^{\frac{d_0}{3}} q_1^{d_1} q_2^{d_2} \frac{-(-1)^{\frac{-d_0 - d_1 - 2d_2}{3} + \lfloor \frac{-fd_0}{3} \rfloor}}{\frac{d_0}{3} \cdot (\frac{d_0 - 2d_1 - d_2}{3})! \cdot d_1! d_2!} \cdot \frac{\Gamma(\frac{(f+1)d_0 + d_1 + 2d_2}{3})}{\Gamma(\frac{fd_0}{3} + 1)} \mathbf{1}_{\omega^{d_1 - d_2}} \end{aligned}$$

4.2.2. A framed inner brane.

Theorem 4.6.

$$(14) \quad W_v^{\mathcal{X},(\mathcal{L},f)}(q, x) = \sum_{\substack{\tilde{\beta}=(d_0, \beta) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \\ v(\tilde{\beta})=v}} q^{\tilde{\beta}} A_{\tilde{\beta}}^{\mathcal{X},(\mathcal{L},f)}.$$

where

$$\begin{aligned} \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) &= \{ \tilde{\beta} = (d_0, \beta) \in (\mathbb{Z} - \{0\}) \times \mathbb{K}_{\text{eff}} : \langle D_i, \tilde{\beta} \rangle \in \mathbb{Z}_{\geq 0} \text{ for } i \in I_\tau \} \\ A_{\tilde{\beta}=(d, \beta)}^{\mathcal{X},(\mathcal{L},f)} &= -(-1)^{\frac{d_0}{s_1^+}(-f_+-1) - \{c_{i_2}(v) - fc_{i_1}(v)\} + \langle D_{i_3}, \beta \rangle} \frac{s_1}{d_0} \left[-\langle D_{i_3}, \tilde{\beta} \rangle - 1 \right] \prod_{i \in I_\tau} \left[\begin{matrix} 0 \\ \langle D_i, \tilde{\beta} \rangle \end{matrix} \right]. \end{aligned}$$

Proof.

$$W_v^{\mathcal{X},(\mathcal{L},f)}(q, x) = I_+ + I_-,$$

where

$$\begin{aligned} I_+ &= \sum_{\substack{d_0 \in \mathbb{Z}_{>0} \\ \langle \frac{d_0}{s_1^+} \rangle = \epsilon_1}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}}, \sigma_+ \\ v(\beta)=v}} -x^{d_0} q^\beta (-1)^{\frac{d_0}{s_1^+}(-f^+-1) - \{\epsilon_1 - f\epsilon_2\} + \langle D_{i_3}, \beta \rangle} \frac{s_1^+}{d_0} \left[\frac{\frac{(f^++1)d_0}{s_1^+} - \langle D_{i_3}, \beta \rangle - 1}{\frac{f^+d_0}{s_1^+} + \langle D_{i_2}, \beta \rangle} \right] \\ &\quad \cdot \left[\begin{matrix} 0 \\ \frac{d_0}{s_1^+} + \langle D_{i_1}, \beta \rangle \end{matrix} \right] \cdot \left[\begin{matrix} 0 \\ \langle D_{i_4}, \beta \rangle \end{matrix} \right] \cdot \prod_{I - \{i_1, i_2, i_3, i_4\}} \left[\begin{matrix} 0 \\ \langle D_i, \beta \rangle \end{matrix} \right] \\ I_- &= \sum_{\substack{d_0 \in \mathbb{Z}_{<0} \\ \langle \frac{-d_0}{s_1^-} \rangle = \epsilon_1}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}}, \sigma_- \\ v(\beta - d_0\alpha)=v}} -x^{d_0} q^{\beta - d_0\alpha} (-1)^{\frac{d_0}{s_1^-}(-f^-1) - \{\epsilon_1 - f\epsilon_2\} - \langle D_{i_2}, \beta \rangle} \frac{s_1^-}{d_0} \left[\frac{\frac{-(f^-+1)d_0}{s_1^-} - \langle D_{i_2}, \beta \rangle - 1}{\frac{-f^-d_0}{s_1^-} + \langle D_{i_3}, \beta \rangle} \right] \\ &\quad \cdot \left[-\frac{d_0}{s_1^-} + \langle D_{i_4}, \beta \rangle \right] \cdot \left[\begin{matrix} 0 \\ \langle D_{i_1}, \beta \rangle \end{matrix} \right] \cdot \prod_{I - \{i_1, i_2, i_3, i_4\}} \left[\begin{matrix} 0 \\ \langle D_i, \beta \rangle \end{matrix} \right] \frac{s_1^+}{s_1^-} \end{aligned}$$

We have

$$\langle D_{i_1}, \alpha \rangle = \frac{1}{s_1^+}, \quad \langle D_{i_2}, \alpha \rangle = \frac{f^+}{s_1^+} - \frac{f^- + 1}{s_1^-}, \quad \langle D_{i_3}, \alpha \rangle = \frac{f^-}{s_1^-} - \frac{f^+ + 1}{s_1^+}, \quad \langle D_{i_4}, \alpha \rangle = \frac{1}{s_1^-},$$

and $\langle D_i, \alpha \rangle = 0$ for $i \in I - \{i_0, i_1, i_3, i_4\}$. So for $\beta \in \mathbb{K}_{\text{eff}, \sigma_-}$,

$$\begin{aligned} \langle D_{i_1}, \beta \rangle &= \langle D_{i_1}, \beta - d_0 \alpha \rangle + \frac{d_0}{s_1^+} \\ -\frac{(f^- + 1)d_0}{s_1^-} - \langle D_{i_2}, \beta \rangle &= -\frac{f^+ d_0}{s_1^+} - \langle D_{i_2}, \beta - d_0 \alpha \rangle \\ -\frac{f^- d_0}{s_1^-} + \langle D_{i_3}, \beta \rangle &= -\frac{(f^+ + 1)d_0}{s_1^+} + \langle D_{i_3}, \beta - d_0 \alpha \rangle \\ -\frac{d_0}{s_1^-} + \langle D_{i_4}, \beta \rangle &= \langle D_{i_4}, \beta - d_0 \alpha \rangle \end{aligned}$$

So

$$\begin{aligned} I_- &= \sum_{\substack{d_0 \in \mathbb{Z}_{<0} \\ \langle \frac{-d_0}{s_1^+} \rangle = \epsilon_1}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}, \sigma_-} \\ v(\beta - d_0 \alpha) = v}} -x^{d_0} q^{\beta - d_0 \alpha} (-1)^{\frac{d_0}{s_1^+}(-f^- - 1) - \{\epsilon_2 - f\epsilon_1\} - \langle D_{i_2}, \beta \rangle} \frac{s_1^+}{d_0} \left[\frac{-\frac{f^+ d_0}{s_1^+} - \langle D_{i_2}, \beta - d_0 \alpha \rangle - 1}{\frac{-(f^+ + 1)d_0}{s_1^+} + \langle D_{i_3}, \beta - d_0 \alpha \rangle} \right] \\ &\quad \cdot \left[\frac{d_0}{s_1^+} + \langle D_{i_1}, \beta - d_0 \alpha \rangle \right] \cdot \left[\langle D_{i_4}, \beta - d_0 \alpha \rangle \right] \cdot \prod_{I - \{i_1, i_2, i_3, i_4\}} \left[\langle D_i, \beta - d_0 \alpha \rangle \right] \\ &= \sum_{\substack{d_0 \in \mathbb{Z}_{<0} \\ \langle \frac{-d_0}{s_1^+} \rangle = \epsilon_1}} \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}, \sigma_-} \\ v(\beta - d_0 \alpha) = v}} -x^{d_0} q^{\beta - d_0 \alpha} (-1)^{\frac{d_0}{s_1^+}(-f^+ - 1) - \{\epsilon_2 - f\epsilon_1\} + \langle D_{i_3}, \beta - d_0 \alpha \rangle} \frac{s_1^+}{d_0} \left[\frac{\frac{(f^+ + 1)d_0}{s_1^+} - \langle D_{i_3}, \beta - d_0 \alpha \rangle - 1}{\frac{f^+ d_0}{s_1^+} + \langle D_{i_2}, \beta - d_0 \alpha \rangle} \right] \\ &\quad \cdot \left[\frac{d_0}{s_1^+} + \langle D_{i_1}, \beta - d_0 \alpha \rangle \right] \cdot \left[\langle D_{i_4}, \beta - d_0 \alpha \rangle \right] \cdot \prod_{I - \{i_1, i_2, i_3, i_4\}} \left[\langle D_i, \beta - d_0 \alpha \rangle \right] \end{aligned}$$

□

4.3. The B-model and the mirror curve. The mirror B-model to the toric Calabi-Yau threefold \mathcal{X} is another non-compact Calabi-Yau hypersurface $Y \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$, constructed as the Hori-Vafa mirror [35]. It contains a distinguished mirror curve $C \subset \{(0, 0)\} \times (\mathbb{C}^*)^2$. We simply state the relevant results in the most elementary way, and refer to [5, 4] for the mirror prediction of the disk amplitudes from Y .

4.3.1. Mirror curve and the prepotential. The mirror curve C is given by the following equations

$$\begin{aligned} x_1 + \cdots + x_r &= 0, \\ \prod_{i=1}^r x_i^{l_i^{(a)}} &= \hat{q}_a, \quad a = 1, \dots, k. \end{aligned}$$

After a change of variable

$$x_{i_1} = \hat{q}_0 y^{-f}, x_{i_2} = y, x_{i_3} = 1,$$

and writing other x_i in terms of x, y , we arrive at an equation

$$(15) \quad F(y, \hat{q}_0, \hat{q}_1, \dots, \hat{q}_k) = 0,$$

which prescribes an curve in $(\mathbb{C}^*)^2$. This is called the *mirror curve* C to the Calabi-Yau three orbifold \mathcal{X} and the Aganagic-Vafa brane (\mathcal{L}, f) . We denote $\hat{q}_0 = x^{s_1}$.

The definition $\mathbb{K}_{\text{eff}, \sigma} = \{\beta \in \mathbb{L}_{\mathbb{Q}} : \langle D_i, \beta \rangle \in \mathbb{Z}_{\geq 0} \text{ for } i \in I_{\sigma}\}$ prompts us to identify $\mathbb{K}_{\text{eff}, \sigma} = \bigoplus_{i=1}^k \mathbb{Z}_{\geq 0} e^k$. We choose $\{p_a\}_{a=1}^k = \{D_i\}_{i \in I_{\sigma}}$ such that $\langle p_a, e^b \rangle = \delta_{ab}$. In particular, we denote $i(a) \in I_{\sigma}$ with $p_a = D_{i(a)}$. We set $p_0 = D_{i_1}$ and $i(0) = i_1$.

Let $(\tilde{p}_{ab})_{0 \leq a, b \leq k} = (\tilde{l}_{i(a)}^{(b)})$, and $(p_{ab})_{1 \leq a, b \leq k} = (\tilde{l}_{i(a)}^{(b)})$. Then we have

$$\tilde{p}_{00} = 1, \quad \tilde{p}_{a0} = 0, \quad \tilde{p}_{0b} = l_{i_1}^{(b)}, \quad \tilde{p}_{ab} = p_{ab}, \quad \text{when } a, b \in \{1, \dots, k\}.$$

Let $(\tilde{p}^{ab})_{0 \leq a, b \leq k}$ be the inverse matrix of $(\tilde{p}_{ab})_{0 \leq a, b \leq k}$, and $(p^{ab})_{1 \leq a, b \leq k}$ be the inverse matrix of $(p_{ab})_{1 \leq a, b \leq k}$. Then

$$\tilde{p}^{00} = 1, \quad \tilde{p}^{a0} = 0, \quad \tilde{p}^{0b} = -\sum_{a=1}^k l_{i_1}^{(a)} p^{ab} = -\langle D_{i_1}, e^b \rangle, \quad \tilde{p}^{ab} = p^{ab}, \quad \text{when } a, b \in \{1, \dots, k\}.$$

Define $\tilde{e}^b = (s_1 \tilde{p}^{0b}, e^b)$. Then $\langle p_a, \tilde{e}^b \rangle = \delta_{ab}$. Let

$$\mathbb{D}_{\text{eff}} = \bigoplus_{a=0}^k \mathbb{Z}_{\geq 0} \tilde{e}^a.$$

It is obvious that $\mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) = \{(d, \beta) \in \mathbb{D}_{\text{eff}} : d \neq 0\}$. Define

$$(16) \quad \tilde{q}_a := \prod_{b=0}^k \hat{q}_b^{\tilde{p}^{ba}}, \quad a = 0, \dots, k; \quad q_a := \prod_{b=1}^k \hat{q}_b^{p^{ba}}, \quad a = 1, \dots, k;$$

$$\tilde{q}_0 = \hat{q}_0 = q_0 = x^{s_1}.$$

Then

$$\tilde{q}_a = \hat{q}_0^{\tilde{p}^{0a}} \prod_{b=1}^k \hat{q}_b^{\tilde{p}^{ba}} = q_a \hat{q}_0^{-\langle D_{i_1}, e^a \rangle}.$$

We have

$$x_{i_2}^{\tilde{l}_{i_2}^{(a)}} x_{i_3}^{\tilde{l}_{i_3}^{(a)}} \prod_{i \in I_\tau} x_i^{\tilde{l}_i^{(a)}} = \hat{q}_a, \quad a = 1, \dots, k.$$

Let $\{y_0, y_1, \dots, y_k\} = \{x_i\}_{i \in I_\tau}$, $x_{i_2} = y$, $x_{i_3} = 1$. Then

$$\prod_{b=0}^k y_b^{\tilde{p}^{ba}} = \hat{q}_a y^{-\tilde{l}_{i_2}^{(a)}}.$$

So

$$y_a = \prod_{b=0}^k (\hat{q}_b y^{\tilde{l}_{i_2}^{(b)}})^{\tilde{p}^{ba}} = \tilde{q}_a y^{-\sum_{b=0}^k \tilde{l}_{i_2}^{(b)} \tilde{p}^{ba}}.$$

Let

$$\epsilon_a = -\sum_{b=0}^k \tilde{l}_{i_2}^{(b)} \tilde{p}^{ba}, \quad a = 0, \dots, k.$$

Then $\epsilon_0 = -f$,

$$\begin{aligned} \epsilon_a &= \sum_{b=1}^k (f l_{i_1}^{(b)} - l_{i_2}^{(b)}) p^{ba} \\ &= f \langle D_{i_1}, e^a \rangle - \langle D_{i_2}, e^a \rangle, \quad a = 1, \dots, k. \end{aligned}$$

The framed mirror curve equation, in terms of y and new variables $\tilde{q}_0, \dots, \tilde{q}_k$ is

$$(17) \quad H(y, \tilde{q}_0, \dots, \tilde{q}_k) = 0,$$

where

$$H(y, \tilde{q}_0, \dots, \tilde{q}_k) = 1 + y + \sum_{a=0}^k y_a = 1 + y + \sum_{a=0}^k \tilde{q}_a y^{\epsilon_a} = 1 + y + \sum_{a=0}^k \tilde{q}_a y^{\epsilon_a}.$$

We call this $H(y, \tilde{q}_0, \dots, \tilde{q}_k)$ *normalized curve equation*. So

$$\frac{\partial H}{\partial y}(y, \tilde{q}_0, \dots, \tilde{q}_k) = 1 + \sum_{a=0}^k \tilde{q}_a \epsilon_a y^{\epsilon_a - 1}.$$

Write $\tilde{q} = (\tilde{q}_0, \dots, \tilde{q}_k)$. Choose a branch of $\log y$ near $y = -1$ such that $\log(-1) = i\pi$. Then $H(y, \tilde{q})$ is holomorphic near $y = -1$, $\tilde{q} = 0$, and

$$H(-1, 0, \dots, 0) = 0, \quad \frac{\partial H}{\partial y}(-1, 0, \dots, 0) = 1 \neq 0.$$

By implicit function theorem, there exists a holomorphic function $h(\tilde{q}_0, \dots, \tilde{q}_k)$ defined in an open neighborhood of $\tilde{q} = 0$, such that $y(0, \dots, 0) = -1$ and $H(y(\tilde{q}), \tilde{q}) = 0$. We conclude:

Lemma 4.7. *The implicit function $y(\tilde{q}_0, \dots, \tilde{q}_k)$ is a power series in $\tilde{q}_0, \dots, \tilde{q}_k$ with constant term -1 , and $\log y$ is a power series in $\tilde{q}_0, \dots, \tilde{q}_k$ with constant term $i\pi$.*

If we write y as a Laurent series in q_0, q_1, \dots, q_k ,⁴ the B-model prepotential W_H is simply given as an anti-derivative:

$$W_H = \int \frac{\log y(x, q_1, \dots, q_k)}{q_0} dq_0.$$

The Laurent series W_H is decomposed as

$$W_H = f(q_1, \dots, q_k) \log q_0 + W_{H, \text{inst}}(x, q_1, \dots, q_k).$$

The *instanton* part $W_{H, \text{inst}}$ is a Laurent series in x, q_1, \dots, q_k , with no degree 0 term in x .

Let $W_v^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k)$ be the pullback of the disk potential $F_{(0,1),v}^{\mathcal{X}, (\mathcal{L}, f)}(Q, X)$ in sector $v \in \text{Box}(\sigma)$ under the open-closed mirror map. Define a character

$$\chi^{\mathcal{L}, f} : \text{Box}(\sigma) = G_\sigma \rightarrow U(1), \quad v \mapsto e^{2\pi\sqrt{-1}(c_{i_2}(v) - f c_{i_1}(v))}.$$

Define

$$W^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k) := \sum_{v \in \text{Box}(\sigma)} W_v^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k) \sqrt{\chi^{\mathcal{L}, f}(v)},$$

where $\sqrt{\chi^{\mathcal{L}, f}(v)} = e^{\pi\sqrt{-1}(c_{i_2}(v) - f c_{i_1}(v))}$. The mirror conjecture for disk amplitudes, proposed in [5, 4], is the following.

Conjecture 4.8 (Aganagic-Vafa, Aganagic-Klemm-Vafa). *The pullback of the B-model prepotential $W^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k)$ of the disk potential $F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}(Q, X)$ under the open-closed mirror map is equal to the instanton part of the B-model super potential:*

$$W^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k) = W_{H, \text{inst}}(x, q_1, \dots, q_k).$$

Remark 4.9. If \mathcal{X} is a smooth variety, then $\chi_0 = 1$, where $v = 0$ is the only element in $\text{Box}(\sigma)$. We get back to the original form of the conjecture in [5, 4].

4.4. Open mirror theorem for disk amplitudes. The solution v to the exponential polynomial equation

$$s_1 e^{r_1 v} + s_2 e^{r_2 v} + \dots + s_k e^{r_k v} - e^v + 1 = 0,$$

around $s_1 = \dots = s_k = 0, v = 0$ is in the following power series form (see Appendix A for a proof)

$$(18) \quad v = \sum_{\substack{n_1, \dots, n_k=0 \\ (n_1, \dots, n_k) \neq 0}}^{\infty} \frac{(r_1 n_1 + \dots + r_k n_k - 1)_{(n_1 + \dots + n_k - 1)}}{n_1! \dots n_k!} s_1^{n_1} \dots s_k^{n_k}.$$

Here we adopt the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} = \begin{cases} a(a-1) \dots (a-n+1), & n > 0; \\ 1, & n = 0; \\ \frac{1}{(a+1) \dots (a-n)}, & n < 0; \end{cases}$$

where $a \in \mathbb{C}$ and $n \in \mathbb{Z}$. Starting from this simple observation, we prove Conjecture 4.8 in this section.

The mirror curve $H(y, \tilde{q}_0, \dots, \tilde{q}_k)$ is

$$1 + y + \sum_{a=0}^k \tilde{q}_a y^{\epsilon_a} = 0.$$

⁴After a change of variables (16), negative powers may appear.

After a change of variable $y = -y'$ and $\tilde{q}_a = (-1)^{-\epsilon_a} \tilde{q}'_a$, it becomes

$$1 - y' + \sum_{a=0}^k \tilde{q}'_a y'^{\epsilon_a} = 0.$$

For $\beta \in \mathbb{K}_\sigma$ and $\tilde{\beta} = (d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, recall the definition of power notations

$$\begin{aligned} q^\beta &= \prod_{a=1}^k q_a^{\langle p_a, \beta \rangle}, & q^{\tilde{\beta}} &= x^{d_0} \prod_{a=1}^k q_a^{\langle p_a, \beta \rangle}, \\ \tilde{q}^{\tilde{\beta}} &= \prod_{a=0}^k \tilde{q}_a^{\langle \tilde{p}_a, \tilde{\beta} \rangle}, & \tilde{q}^{\tilde{\beta}} &= \prod_{a=0}^k \tilde{q}_a^{\langle \tilde{p}_a, \tilde{\beta} \rangle}. \end{aligned}$$

Hence

$$\begin{aligned} \log y' &= \sum_{\substack{\tilde{d}_0, \dots, \tilde{d}_k=0 \\ (d_0, \dots, d_k) \neq 0}}^{\infty} \frac{(\epsilon_0 \tilde{d}_0 + \dots + \epsilon_k \tilde{d}_k - 1)_{(\tilde{d}_0 + \dots + \tilde{d}_k - 1)}}{\tilde{d}_0! \dots \tilde{d}_k!} \tilde{q}_0^{\tilde{d}_0} \dots \tilde{q}_k^{\tilde{d}_k} \\ &= \sum_{\tilde{\beta}=(d_0, \beta) \in \mathbb{D}_{\text{eff}}, \tilde{\beta} \neq 0} \frac{(-f \langle D_{i_1}, \tilde{\beta} \rangle + f \langle D_{i_1}, \beta \rangle - \langle D_{i_2}, \beta \rangle - 1)_{(\sum_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle - 1)}}{\prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} \tilde{q}^{\tilde{\beta}} \\ &= \sum_{\tilde{\beta} \in \mathbb{D}_{\text{eff}}, \tilde{\beta} \neq 0} \frac{(-\langle D_{i_2}, \tilde{\beta} \rangle - 1)_{(-\langle D_{i_2}, \tilde{\beta} \rangle - \langle D_{i_3}, \tilde{\beta} \rangle - 1)}}{\prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} \tilde{q}^{\tilde{\beta}} \\ &= \sum_{\tilde{\beta} \in \mathbb{D}_{\text{eff}}, \tilde{\beta} \neq 0} (-1)^{-\langle D_{i_2}, \tilde{\beta} \rangle} \frac{(-\langle D_{i_2}, \tilde{\beta} \rangle - 1)_{(-\langle D_{i_2}, \tilde{\beta} \rangle - \langle D_{i_3}, \tilde{\beta} \rangle - 1)}}{\prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} \tilde{q}^{\tilde{\beta}} \\ &= \sum_{\tilde{\beta} \in \mathbb{D}_{\text{eff}}, \tilde{\beta} \neq 0} (-1)^{\langle D_{i_3}, \tilde{\beta} \rangle + 1} \frac{(-\langle D_{i_3}, \tilde{\beta} \rangle - 1)_{(-\langle D_{i_2}, \tilde{\beta} \rangle - \langle D_{i_3}, \tilde{\beta} \rangle - 1)}}{\prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} x^{d_0} q^\beta. \end{aligned}$$

It follows that

$$\begin{aligned} \int \frac{\log y}{q_0} dq_0 &= f(q_1, \dots, q_n) \log q_0 \\ &+ \sum_{\tilde{\beta}=(d_0, \beta) \in \mathbb{D}_{\text{eff}}, d_0 \neq 0} (-1)^{\langle D_{i_3}, \tilde{\beta} \rangle + 1} \frac{(-\langle D_{i_2}, \tilde{\beta} \rangle - 1)_{(-\langle D_{i_2}, \tilde{\beta} \rangle - \langle D_{i_3}, \tilde{\beta} \rangle - 1)}}{\frac{d_0}{s_1} \prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} x^{d_0} q^\beta. \end{aligned}$$

We conclude that

Lemma 4.10.

(19)

$$W_{H, \text{inst}}(x, q_1, \dots, q_k) = \sum_{\tilde{\beta}=(d_0, \beta) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L})} (-1)^{\langle D_{i_3}, \beta \rangle - \frac{(f+1)d_0}{s_1} + 1} \frac{(-\langle D_{i_2}, \tilde{\beta} \rangle - 1)_{(-\langle D_{i_2}, \tilde{\beta} \rangle - \langle D_{i_3}, \tilde{\beta} \rangle - 1)}}{\frac{d_0}{s_1} \prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!} x^{d_0} q^\beta.$$

In Theorems 4.4 and 4.6 the pulled-back disk amplitude

$$W_v^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k) = \sum_{\substack{\tilde{\beta}=(d_0, \beta) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \\ v=v(\beta)}} x^{d_0} q^\beta A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)},$$

where

$$(20) \quad A_{\tilde{\beta}}^{\mathcal{X}, (\mathcal{L}, f)} = (-1)^{\langle D_{i_3}, \beta \rangle - \frac{d_0}{s_1} (f+1) + 1 - \{c_{i_2}(v) - f c_{i_1}(v)\}} \frac{(-\langle D_{i_2}, \tilde{\beta} \rangle - 1)_{(-\langle D_{i_2}, \tilde{\beta} \rangle - \langle D_{i_3}, \tilde{\beta} \rangle - 1)}}{\frac{d_0}{s_1} \prod_{i \in I_\tau} \langle D_i, \tilde{\beta} \rangle!}.$$

It follows that

$$W_{H,\text{inst}}(x, q_1, \dots, q_k) = \sum_{v \in \text{Box}(\sigma)} W_v^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1, \dots, q_k) \sqrt{\chi^{\mathcal{L}, f}(v)}.$$

This gives Conjecture 4.8.

Example 4.11. $\mathcal{X} = \mathcal{X}_{1,1,1} = [\mathbb{C}^3/\mathbb{Z}_3]$, $s_1 = 3$. The framed mirror curve is

$$x^3 y^{-f} + y + 1 + \hat{q}_1^{-\frac{1}{3}} x y^{\frac{1-f}{3}} = 0.$$

After a change of variables

$$\tilde{q}_0 = \hat{q}_0 = x^3, \quad \tilde{q}_1 = \hat{q}_1^{-\frac{1}{3}} \hat{q}_0^{\frac{1}{3}},$$

the curve equation becomes

$$\tilde{q}_0 y^{-f} + \tilde{q}_1 y^{\frac{1-f}{3}} + y + 1 = 0.$$

The B-model pre-potential

$$\log y = \sum_{\tilde{d}_0, \tilde{d}_1 \geq 0} \frac{(-f\tilde{d}_0 + \frac{1-f}{3}\tilde{d}_1 - 1)_{(\tilde{d}_0 + \tilde{d}_1 - 1)}}{\tilde{d}_0! \tilde{d}_1!} \tilde{q}_0^{\tilde{d}_0} \tilde{q}_1^{\tilde{d}_1}.$$

Let $d_0 = 3\tilde{d}_0 + \tilde{d}_1$ and $d_1 = \tilde{d}_1$.

$$W_{H,\text{inst}} = \sum_{\substack{d_0 \in \mathbb{Z}_{>0}, d_1 \in \mathbb{Z}_{\geq 0} \\ d_0 - d_1 \in 3\mathbb{Z}_{\geq 0}}} (-1)^{-\frac{(f+1)d_0}{3} - \frac{d_1}{3}} \frac{1}{\frac{d_0}{3}(\frac{d_0-d_1}{3})! d_1!} \cdot \frac{\Gamma(\frac{f+1}{3}d_0 + \frac{d_1}{3})}{\Gamma(\frac{f d_0 - d_1}{3} + 1)} x^d q_1^{d_1}.$$

Setting $\chi_v = -e^{-\sqrt{-1}\pi(\lfloor \frac{(f-1)d}{3} \rfloor + (f+4)\{\frac{d}{3}\})}$,

$$W_{H,\text{inst}}(x, q_1) = \sum_{v \in \text{Box}(\sigma)} W_v^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1) \sqrt{\chi^{\mathcal{L}, f}(v)},$$

where the pullback prepotential $W_v^{\mathcal{X}, (\mathcal{L}, f)}(x, q_1)$ in the twisted-sector v is given in Example 4.5, and

$$\sqrt{\chi^{\mathcal{L}, f}(1)} = 1, \quad \sqrt{\chi^{\mathcal{L}, f}(e^{2\pi\sqrt{-1}/3})} = e^{\pi\sqrt{-1}(1-f)/3}, \quad \sqrt{\chi^{\mathcal{L}, f}(e^{4\pi\sqrt{-1}/3})} = e^{2\pi\sqrt{-1}(1-f)/3},$$

Remark 4.12. The framed mirror curve of an outer brane in $K_{\mathbb{P}^2}$, the crepant resolution of the coarse moduli space of $\mathcal{X}_{1,1,1}$, is

$$(21) \quad \hat{q} \hat{x}^3 \hat{y}^{-3\hat{f}-1} + y + 1 + \hat{x} y^{-\hat{f}} = 0,$$

or equivalently,

$$(22) \quad y^{3\hat{f}+2} + y^{3\hat{f}+1} + \hat{q} \hat{x}^3 + \hat{x} y^{2\hat{f}+1} = 0.$$

Equation (22) coincides with Equation (3.19) in [8]. The framed mirror curve of $\mathcal{X}_{1,1,1}$ is

$$(23) \quad x^3 y^{-f} + y + 1 + q_{\text{orb}} x y^{\frac{1-f}{3}} = 0.$$

(21) and (23) are equivalent under the following change of B-model coordinates and framing.

$$q_{\text{orb}} = \hat{q}^{-\frac{1}{3}}, \quad x = \hat{x} \hat{q}^{\frac{1}{3}}, \quad f = 3\hat{f} + 1.$$

The change of coordinates

$$(q_{\text{orb}}, x) \mapsto (q_{\text{orb}}^{-3}, x q_{\text{orb}}^{-1}) = (\hat{q}, \hat{x})$$

is a 3-to-1 map, and the inverse map

$$(\hat{q}, \hat{x}) \mapsto (\hat{q}^{-\frac{1}{3}}, \hat{x} \hat{q}^{\frac{1}{3}}) = (q_{\text{orb}}, x)$$

is multi-valued.

Example 4.13. $\mathcal{X} = \mathcal{X}_{0,1,2} = [\mathbb{C}^2/\mathbb{Z}_3] \times \mathbb{C}$, $s_1 = 1$. The framed mirror curve is

$$xy^{-f} + y + 1 + \hat{q}_1^{-\frac{2}{3}} \hat{q}_2^{-\frac{1}{3}} y^{\frac{1}{3}} + \hat{q}_1^{-\frac{1}{3}} \hat{q}_2^{-\frac{2}{3}} y^{\frac{2}{3}} = 0.$$

After a change of variables

$$\tilde{q}_0 = \hat{q}_0 = x, \quad \tilde{q}_1 = \hat{q}_1^{-\frac{2}{3}} \hat{q}_2^{-\frac{1}{3}}, \quad \tilde{q}_2 = \hat{q}_1^{-\frac{1}{3}} \hat{q}_2^{-\frac{2}{3}},$$

the curve equation becomes

$$\tilde{q}_0 y^{-f} + \tilde{q}_1 y^{\frac{1}{3}} + \tilde{q}_2 y^{\frac{2}{3}} + y + 1 = 0.$$

The B-model superpotential

$$\log y = \sum_{\tilde{d}_0, \tilde{d}_1, \tilde{d}_2 \geq 0} \frac{(-f\tilde{d}_0 + \frac{1}{3}\tilde{d}_1 + \frac{2}{3}\tilde{d}_2 - 1)_{(\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_2 - 1)}}{\tilde{d}_0! \tilde{d}_1! \tilde{d}_2!} \tilde{q}_0^{\tilde{d}_0} \tilde{q}_1^{\tilde{d}_1} \tilde{q}_2^{\tilde{d}_2}.$$

Let $d_0 = \tilde{d}_0$, $d_1 = \tilde{d}_1$ and $d_2 = \tilde{d}_2$.

$$\begin{aligned} W_{H,\text{inst}} &= \sum_{\substack{d \in \mathbb{Z}_{\geq 0} \\ d_1, d_2 \in \mathbb{Z}_{\geq 0}}} \frac{(-1)^{-fd_0 - \frac{2d_1+d_2}{3}}}{d_0 d_0! d_1! d_2!} \cdot \frac{\Gamma((f+1)d_0 + \frac{2d_1+d_2}{3})}{\Gamma(fd_0 - \frac{d_1+2d_2}{3} + 1)} x_0^d q_1^{d_1} q_2^{d_2}. \end{aligned}$$

$$W_{H,\text{inst}} = \sum_{v \in \text{Box}(\sigma)} W_v^{\mathcal{X},(\mathcal{L},f)}(x, q_1, q_2) \sqrt{\chi^{\mathcal{L},f}(v)}$$

where the pullback prepotential $W_v^{\mathcal{X},(\mathcal{L},f)}(x, q_1, q_2)$ in the twisted-sector v is given in Example 4.5, and

$$\sqrt{\chi^{\mathcal{L},f}(1)} = 1, \quad \sqrt{\chi^{\mathcal{L},f}(e^{2\pi\sqrt{-1}/3})} = e^{\pi\sqrt{-1}/3}, \quad \sqrt{\chi^{\mathcal{L},f}(e^{4\pi\sqrt{-1}/3})} = e^{2\pi\sqrt{-1}/3}.$$

APPENDIX A. PROOF OF EQUATION (18)

The solution to the following exponential polynomial

$$(24) \quad 1 - e^{v'} + \tilde{q}_0 e^{r_0 v'} + \tilde{q}_1 e^{r_1 v'} + \cdots + \tilde{q}_k e^{r_k v'} = 0$$

at $\tilde{q}_0 = \cdots = \tilde{q}_k = 0$ is indeed obtained by solving a system of GKZ-type Picard-Fuchs equations. The issue here is that since we are dealing with orbifolds, the parameters r_0, \dots, r_k might be rational numbers. We minimize our argument by quoting known results for integer parameters. Consider a system of *extended charge vectors*

$$(\tilde{l}_i^{(a)})_{i=1, \dots, k+2}^{a=0, \dots, k} = \begin{pmatrix} 1 & -r_0 & r_0 - 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & -r_1 & r_1 - 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -r_2 & r_2 - 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -r_3 & r_3 - 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -r_k & r_k - 1 & 0 & 0 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix}$$

Assume these charge vectors consist of integers, i.e. $r_a \in \mathbb{Z}$, $a = 0, \dots, k$. As a set of extended charge vectors, these charge vectors prescribe a B-model as a threefold

$$Y = \{(z, w, x, y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2, zw = H(x, y, q_1, \dots, q_k)\}$$

where $H(x, y) = x_1 + \cdots + x_k$ is given by the following equations

$$\begin{aligned} \prod_{i=1}^k x_i^{\tilde{l}_i^{(a)}} &= q_a, \quad a = 1, \dots, k, \\ x_1 &= -x^{r_1}, \quad x_2 = y = e^v, \quad x_3 = 1. \end{aligned}$$

Define two 2-cycles \mathcal{C}_* given by

$$w = 0 = H(x, y), \quad x = x_*, \quad y = y_*.$$

This definition involves a choice of x_* and y_* with $H(x_*, y_*) = 0$. Define another 2-cycle \mathcal{C}_x given by

$$w = 0 = F(x(|z|), y(|z|)), \quad x(0) = x, \\ x(|z|) = x_*, \quad y(|z|) = y_*, \quad \text{when } |z| > \Lambda \text{ for some } \Lambda > 0,$$

The 2-cycle \mathcal{C}_x together with $x(|z|), y(|z|)$ as functions in $|z|$ are regarded as 2-branes wrapping over \mathcal{C}_x in physics literature [5, 4]. These two 2-cycles \mathcal{C}_* and \mathcal{C}_x bound a compact 3-cycle $\Gamma(x)$, i.e. $\partial\Gamma = \mathcal{C}_x - \mathcal{C}_*$. We denote that the chain integral, which does not depend on the choice of $\Gamma(x)$, to be

$$P = \frac{1}{2\pi i} \int_{\Gamma(x)} \Omega.$$

Lerche and Mayr [40] show that the 3-chain integral P prescribed by this system of charge vectors are annihilated by the following Picard-Fuchs operators ($q_0 = -x$)

$$\mathcal{D}_a = \prod_{\tilde{l}_i^{(a)} > 0} \prod_{j=0}^{\tilde{l}_i^{(a)}-1} \left(\sum_{b=0}^k q_b \frac{\partial}{\partial q_b} - j \right) - q_a \prod_{\tilde{l}_i^{(a)} < 0} \prod_{j=0}^{-\tilde{l}_i^{(a)}-1} \left(\sum_{b=0}^k q_b \frac{\partial}{\partial q_b} - j \right), \quad a = 0, \dots, k.$$

Solving these Picard-Fuchs equations $\mathcal{D}_a P = 0$, $a = 0, \dots, k$

P = double logarithm terms

$$+ \sum_{(d_0, \dots, d_k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^k} \frac{(-1)^{d_0+r_0d_0+\dots+r_kd_k} (r_0d_0 + \dots + r_kd_k - 1)_{(d_1+\dots+d_k-1)}}{d_0 \cdot d_0!d_1! \dots d_k!} x^{d_0} q_1^{d_1} \dots q_k^{d_k}.$$

Notice that Lerche and Mayr do not assume the charge vectors prescribe a toric variety. Their argument is completely on the B-model side. On the other hand, the power series part of this three chain integral is obtained by solving the spectral curve $H(x, y) = 0$, as shown by Aganagic and Vafa [5], i.e.

$$\int v \frac{dx}{x} = \text{logarithm terms} + \text{power series part of } P.$$

Given these charge vectors, the spectral curve equation $H(x, y) = 0$ is

$$1 + e^v - x e^{r_0 v} + q_1 e^{r_1 v} + q_k e^{r_k v} + \dots + q_k e^{r_k v} = 0.$$

Hence the explicit form of the power series part of P implies

$$v = \sqrt{-1}\pi + f_k(q_1, \dots, q_k) \\ + \sum_{(d_0, \dots, d_k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^k} \frac{(-1)^{d_0+r_0d_0+\dots+r_kd_k} (r_0d_0 + \dots + r_kd_k - 1)_{(d_1+\dots+d_k-1)}}{d_0!d_1! \dots d_k!} x^{d_0} q_1^{d_1} \dots q_k^{d_k}.$$

where $f(q_1, \dots, q_k)$ is an analytic function with $f(0) = 0$. After a change of variables

$$y = -y' = -e^{v'}, \quad v' = v - \sqrt{-1}\pi, \quad x = -(-1)^{-r_0} \tilde{q}_0', \\ q_a = (-1)^{-r_a} \tilde{q}_a', \quad a = 1, \dots, k,$$

the curve equation becomes

$$1 - e^{v'} + \tilde{q}_0 e^{r_0 v'} + \tilde{q}_1 e^{r_1 v'} + \dots + \tilde{q}_k e^{r_k v'} = 0,$$

The solution becomes

$$v' = f'_k(\tilde{q}_1, \dots, \tilde{q}_k) + \sum_{(d_0, \dots, d_k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^k} \frac{(r_0n_0 + \dots + r_kn_k - 1)_{(d_1+\dots+d_k-1)}}{d_0!d_1! \dots d_k!} \tilde{q}_0'^{d_0} \tilde{q}_1'^{d_1} \dots \tilde{q}_k'^{d_k}.$$

Setting $\tilde{q}_0 = 0$, $v'(0, \tilde{q}_1, \dots, \tilde{q}_k) = f'_k(\tilde{q}_1, \dots, \tilde{q}_k)$ is again the solution to the Equation (24) in the same form with one less variable. Repeating the same argument, we have

$$f'_k = f'_{k-1}(\tilde{q}_2, \dots, \tilde{q}_{k-1}) + \sum_{(d_1, \dots, d_k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^{k-1}} \frac{(r_1d_1 + \dots + r_kd_k - 1)_{(d_1+\dots+d_k-1)}}{d_1!d_2! \dots d_k!} \tilde{q}_1'^{d_1} \tilde{q}_2'^{d_2} \dots \tilde{q}_k'^{d_k}.$$

By induction and the observation that $v'(0, \dots, 0) = 0$, we have

$$v' = \sum_{\substack{d_1, \dots, d_k=0 \\ (d_1, \dots, d_k) \neq 0}}^{\infty} \frac{(r_0 d_0 + \dots + r_k d_k - 1)_{(d_1 + \dots + d_k - 1)}}{d_0! d_1! \dots d_k!} \tilde{q}_0^{d_0} \tilde{q}_1^{d_1} \dots \tilde{q}_k^{d_k}.$$

Let

$$v' = \sum_{\substack{d_1, \dots, d_k=0 \\ (d_1, \dots, d_k) \neq 0}}^{\infty} c_{d_0, \dots, d_k} \tilde{q}_0^{d_0} \dots \tilde{q}_k^{d_k}$$

be the power series solution to the Equation (24). We conclude that

$$c_{d_0, \dots, d_k} = \frac{(r_0 d_0 + \dots + r_k d_k - 1)_{(d_1 + \dots + d_k - 1)}}{d_0! d_1! \dots d_k!}$$

when $r_0, \dots, r_k \in \mathbb{Z}$. Since c_{d_0, \dots, d_k} is a rational function of r_0, \dots, r_k , this expression has to be true for all complex-valued r_a .

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